CPSC 490 – Problem Solving in Computer Science

Lecture 13: Segment Tree

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Based on slides by Paul Liu (2014), Kuba Karpierz and Bruno Vacherot (2015)

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Last Time

Last time: SQRT decomposition for range queries

- **Online** square root decomposition can do online point update and range query in $O(\sqrt{n})$ per query
- **Offline** square root decomposition of queries gives amortized $O(\sqrt{n})$ instead of $O(n)$ per query, assuming $q \geq n$.

Today's goal: improve online queries to $O(\log n)$

Range Query Again

Given an array A of n integers, answer these queries quickly:

- `update(i, x)`: sets `A[i] = x`
- `sum(i, j)`: return `A[i] + ... + A[j]`

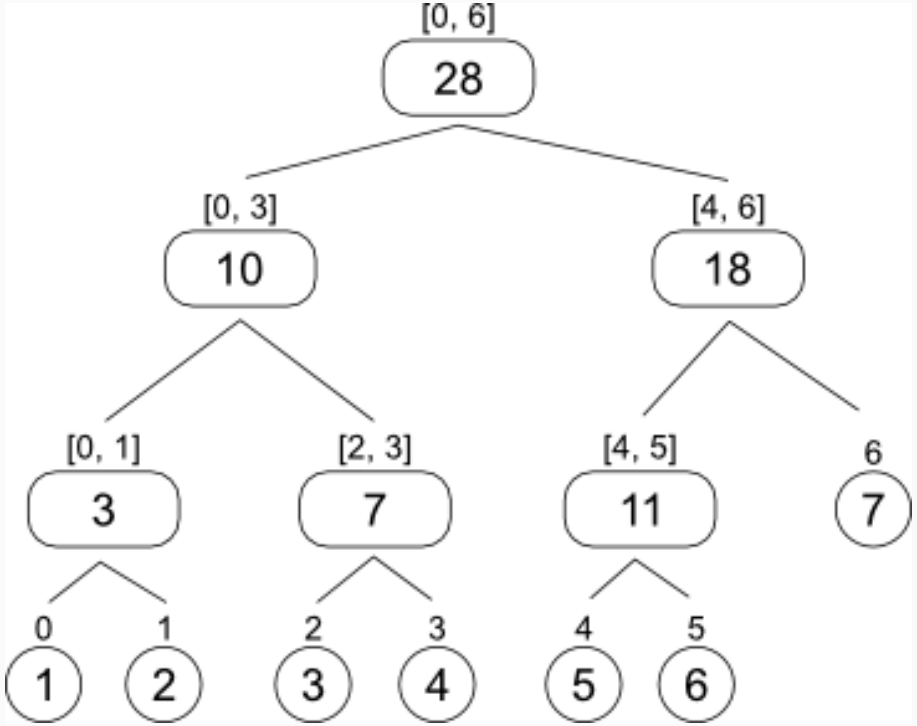
Square root decomposition solves this in $O(\sqrt{n})$ per query.

Segment tree can improve this to $O(\log n)$

Segment Trees

Basic idea: add a lot more layers!

- Square root decomposition: group every $\sqrt{n}$ elements
- Segment tree: group every pair together, then every pair of pairs, then every pair of that, until you get 1 node



Source: `http://scvalex.github.io/articles/SegmentTree.html`

Segment Trees – Structure, Update

Structure

- Binary tree of $\log n$ layers, $1 + 2 + 4 + \cdots + n = O(n)$ nodes
- Leaf node: represent one element
- Internal node: represent union of interval of left + right child

Point Update

- Update leaf
- Update ancestors

Segment Trees (Sum Query) – Structure, Update

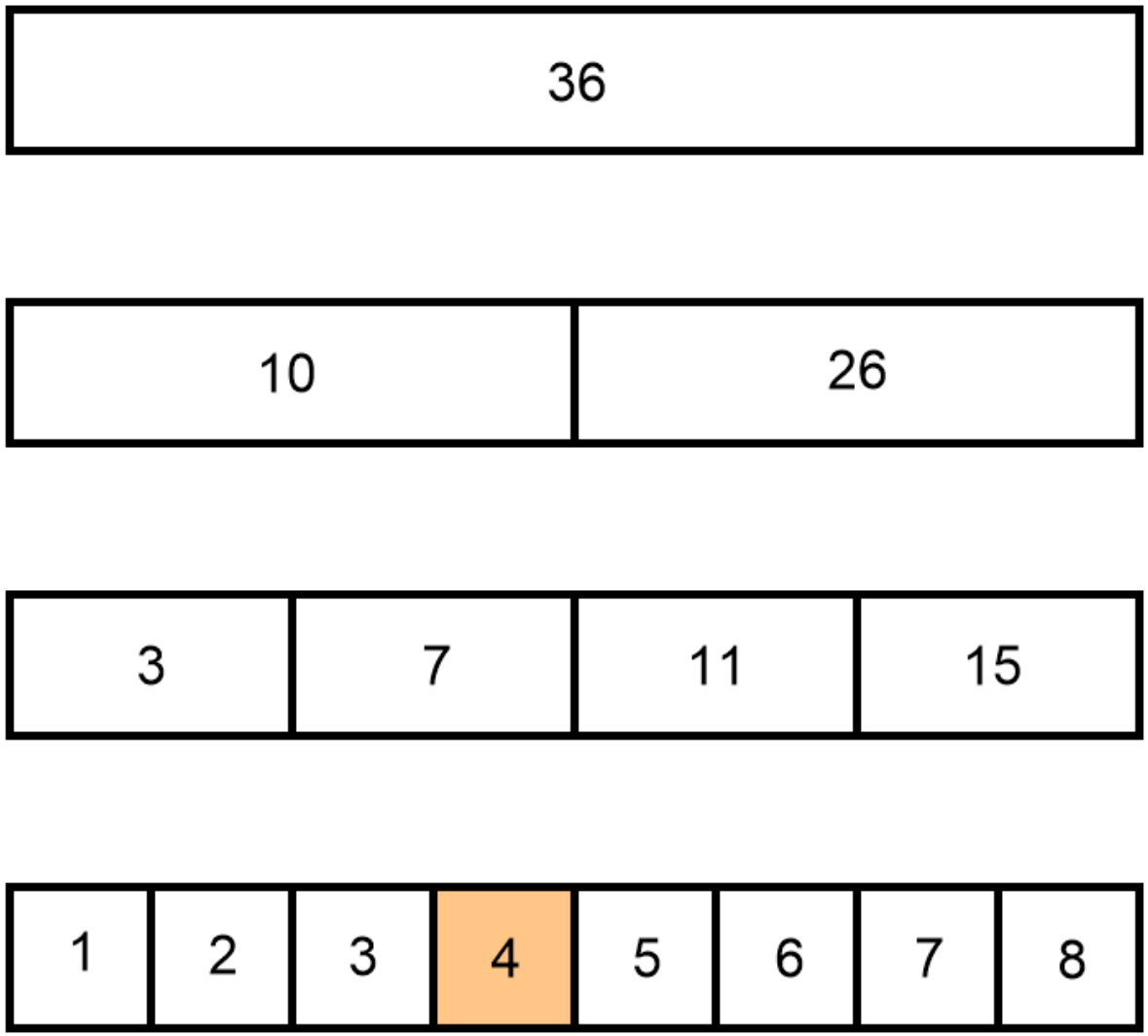
Structure

- Binary tree of $\log n$ layers, $1 + 2 + 4 + \cdots + n = O(n)$ nodes
- Leaf node: store **value of one element**
- Internal node: store **sum of left + right child**

Point Update

- Update leaf: **change value of leaf node**
- Update ancestor: **re-compute sum of left + right child**

Segment Trees (Sum Query) – Point Update



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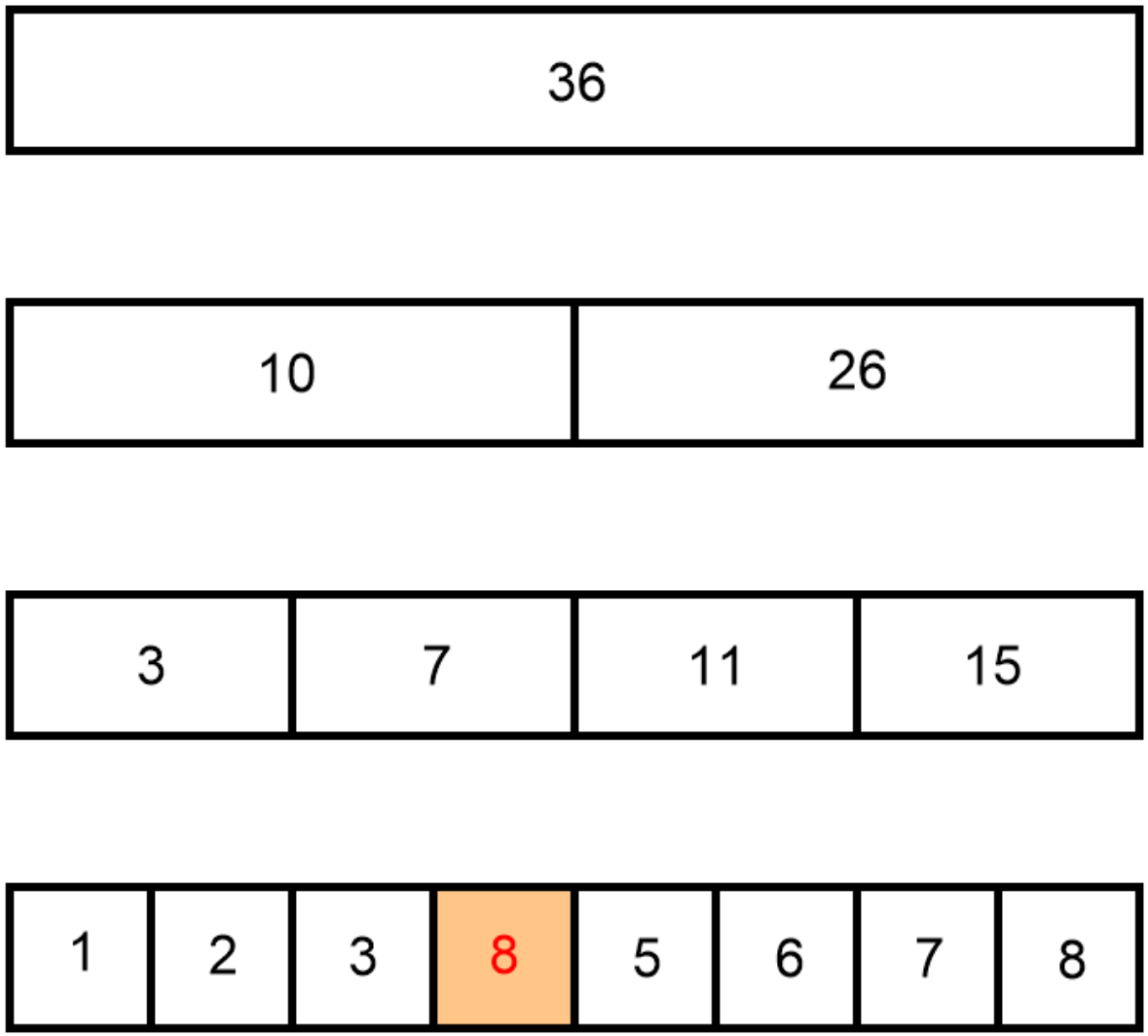

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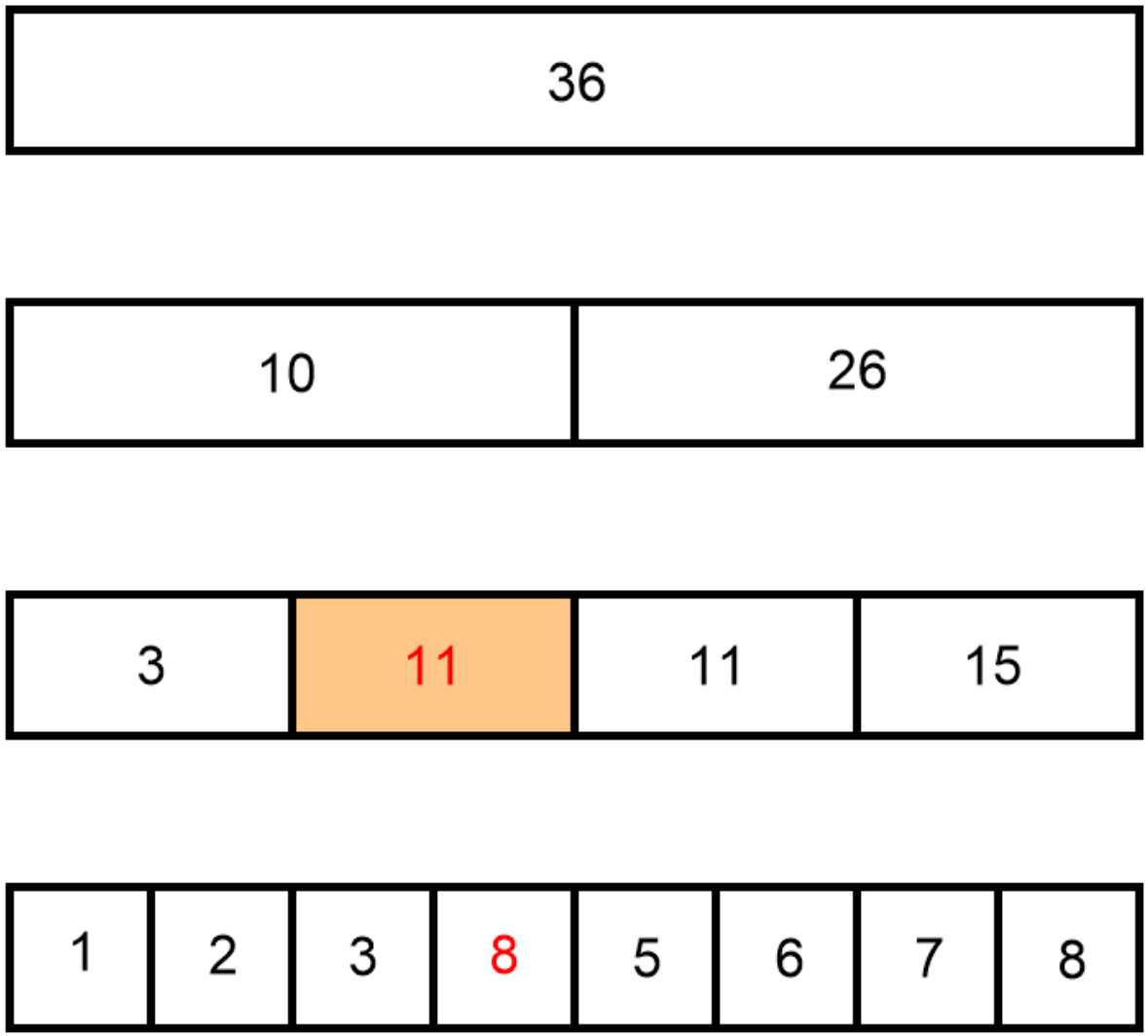

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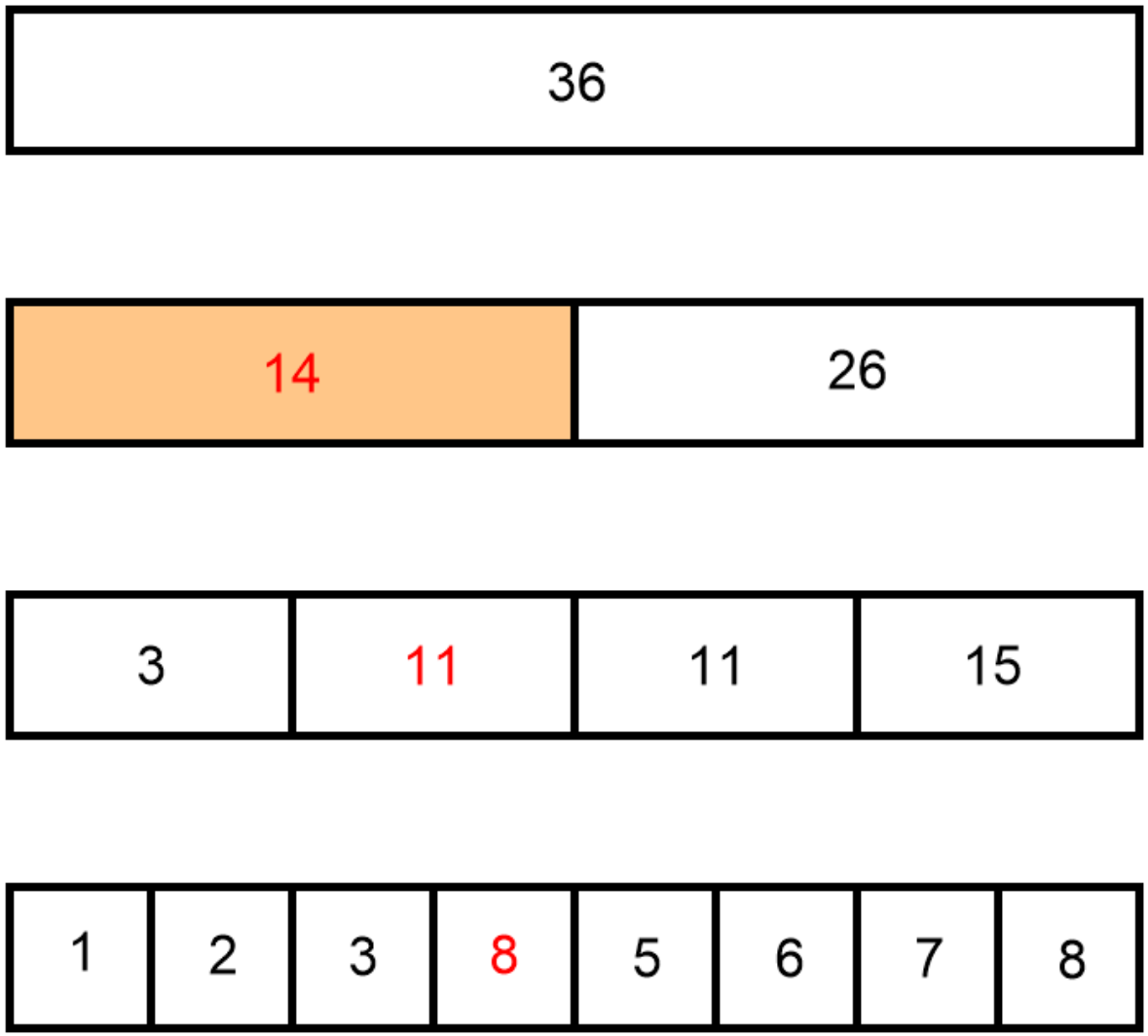

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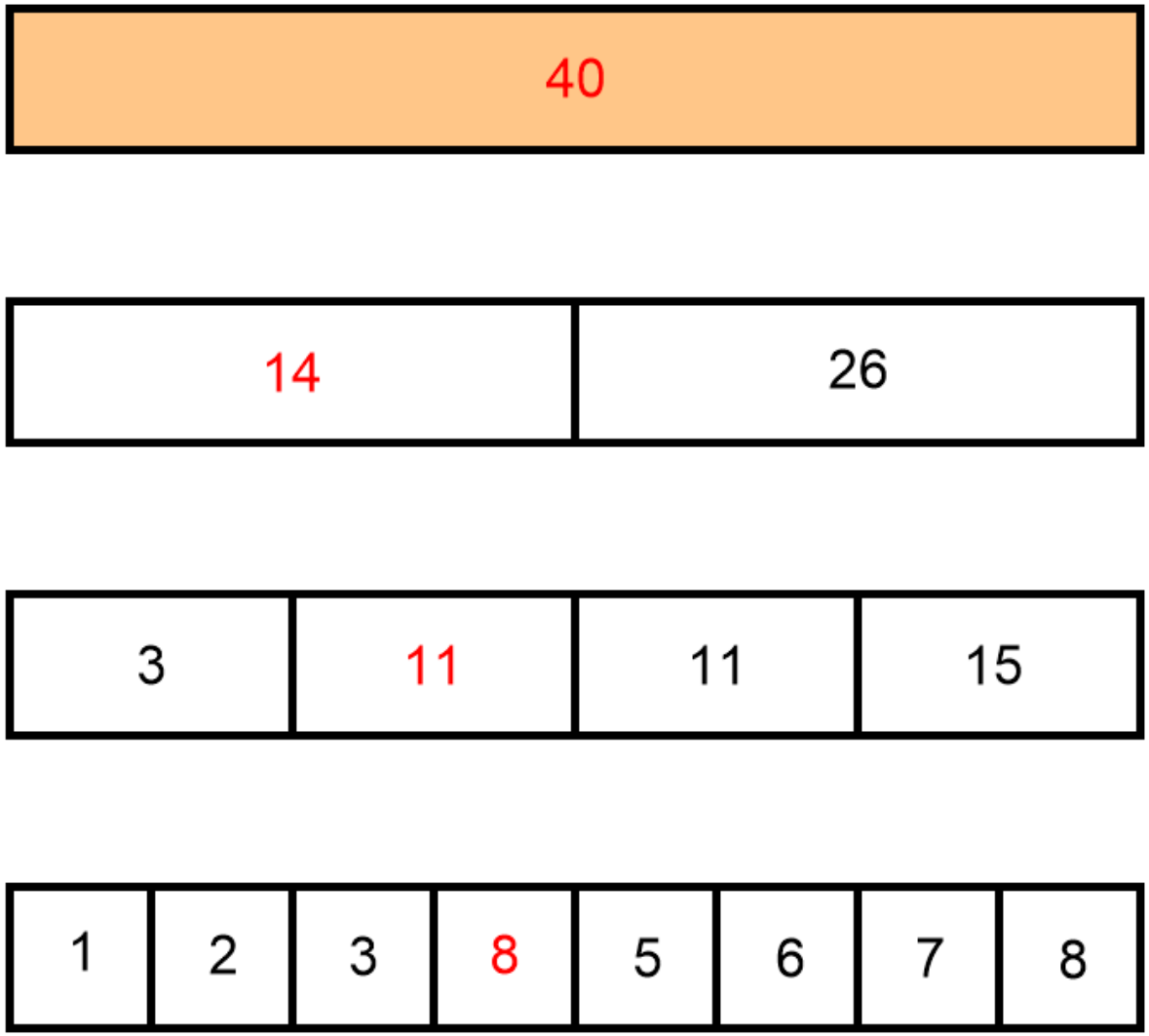

Segment Trees – Range Query

Range query

- Start at root and recurse down
- Suppose query for $[a, b]$, current node represent $[l, r]$
- Case 1: $[l, r] \subseteq [a, b]$, return value of current node
- Case 2: $[l, r] \not\subseteq [a, b]$, recurse then combine answer from children
  1. If $[l, m] \cap [a, b] \neq \emptyset$ recurse left child and get answer
  2. If $[m + 1, r] \cap [a, b] \neq \emptyset$ recurse right child and get answer
  3. Combine above two answers

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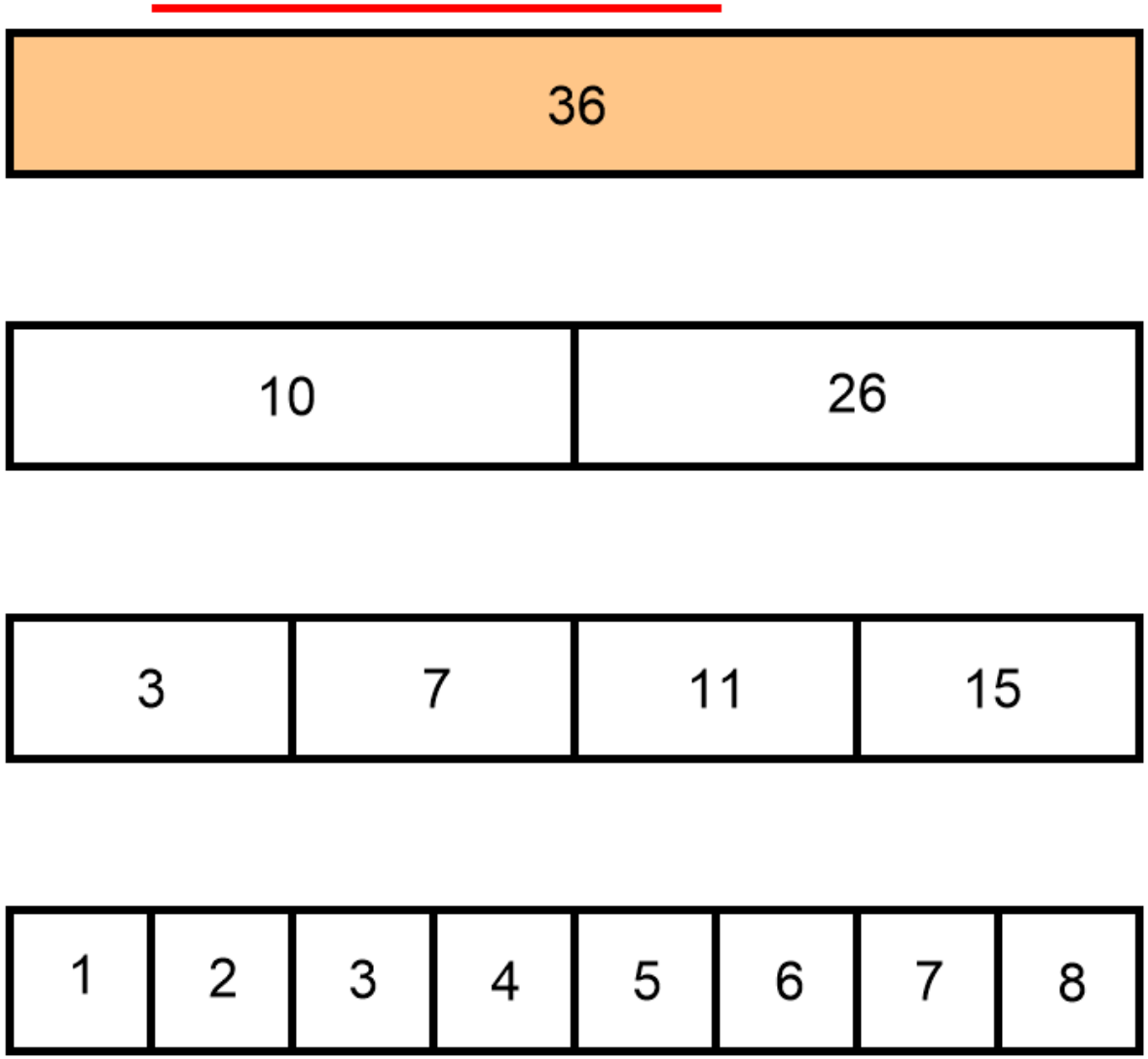
     Sum query: add #1 and #2

Segment Trees (Sum Query) – Range Query



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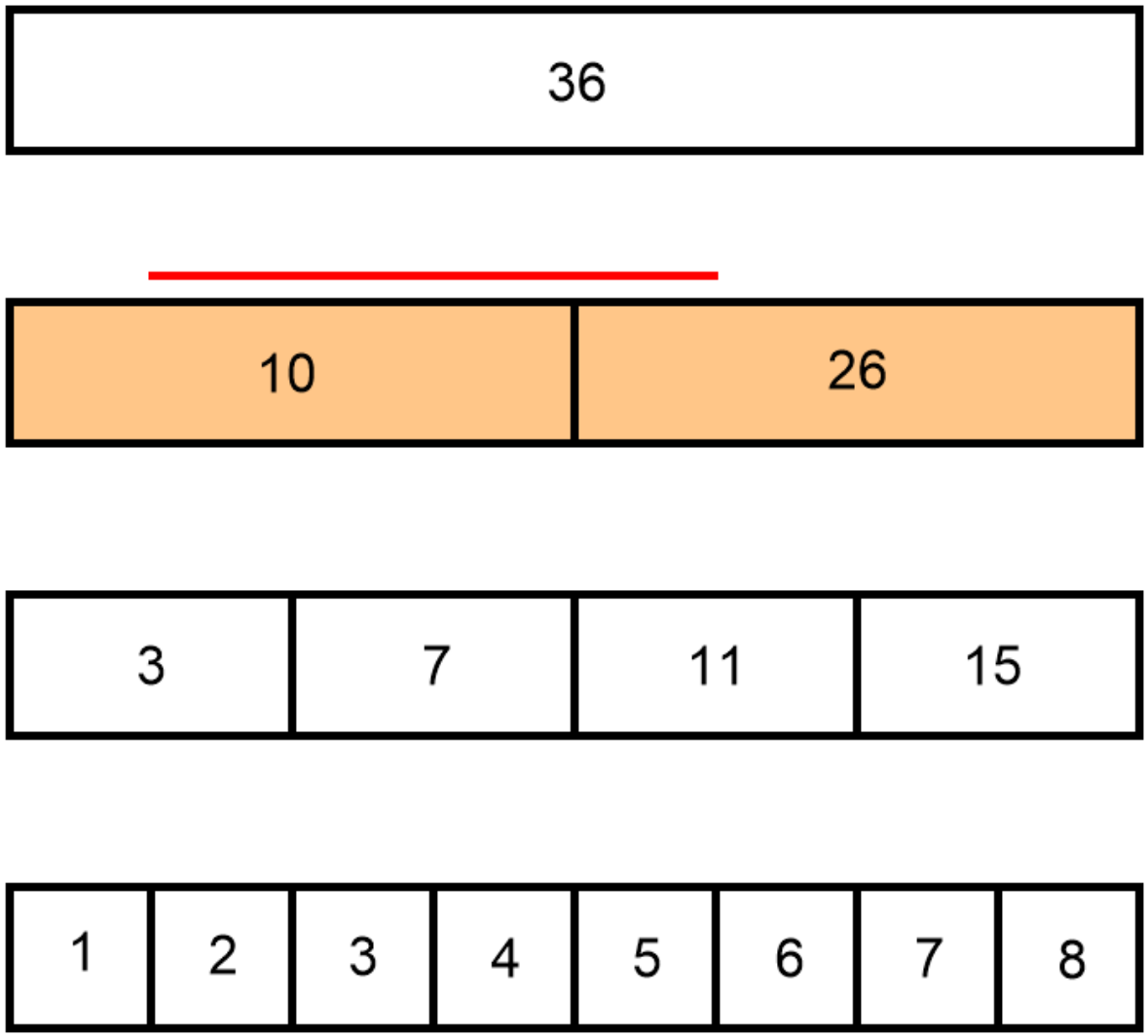

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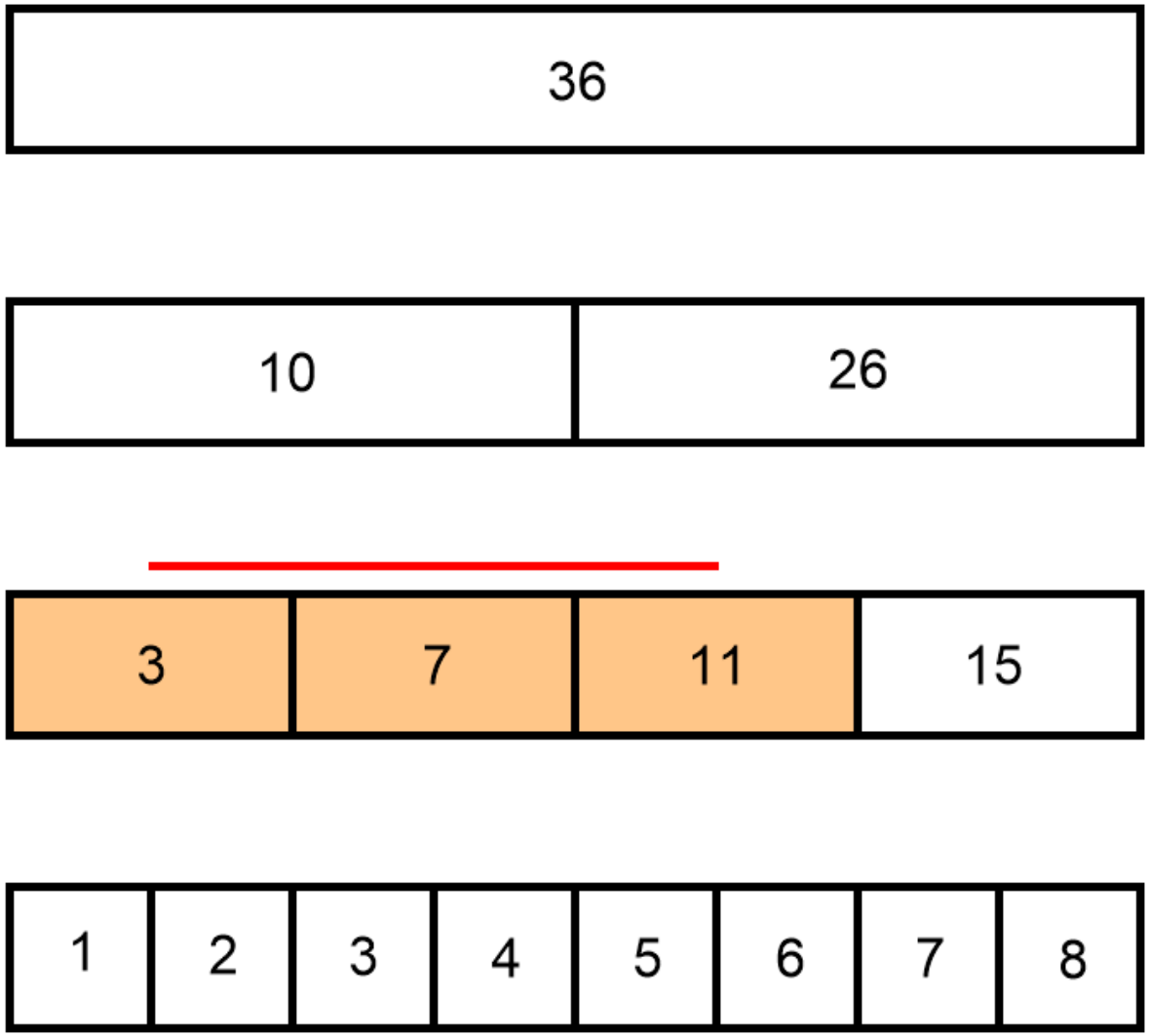

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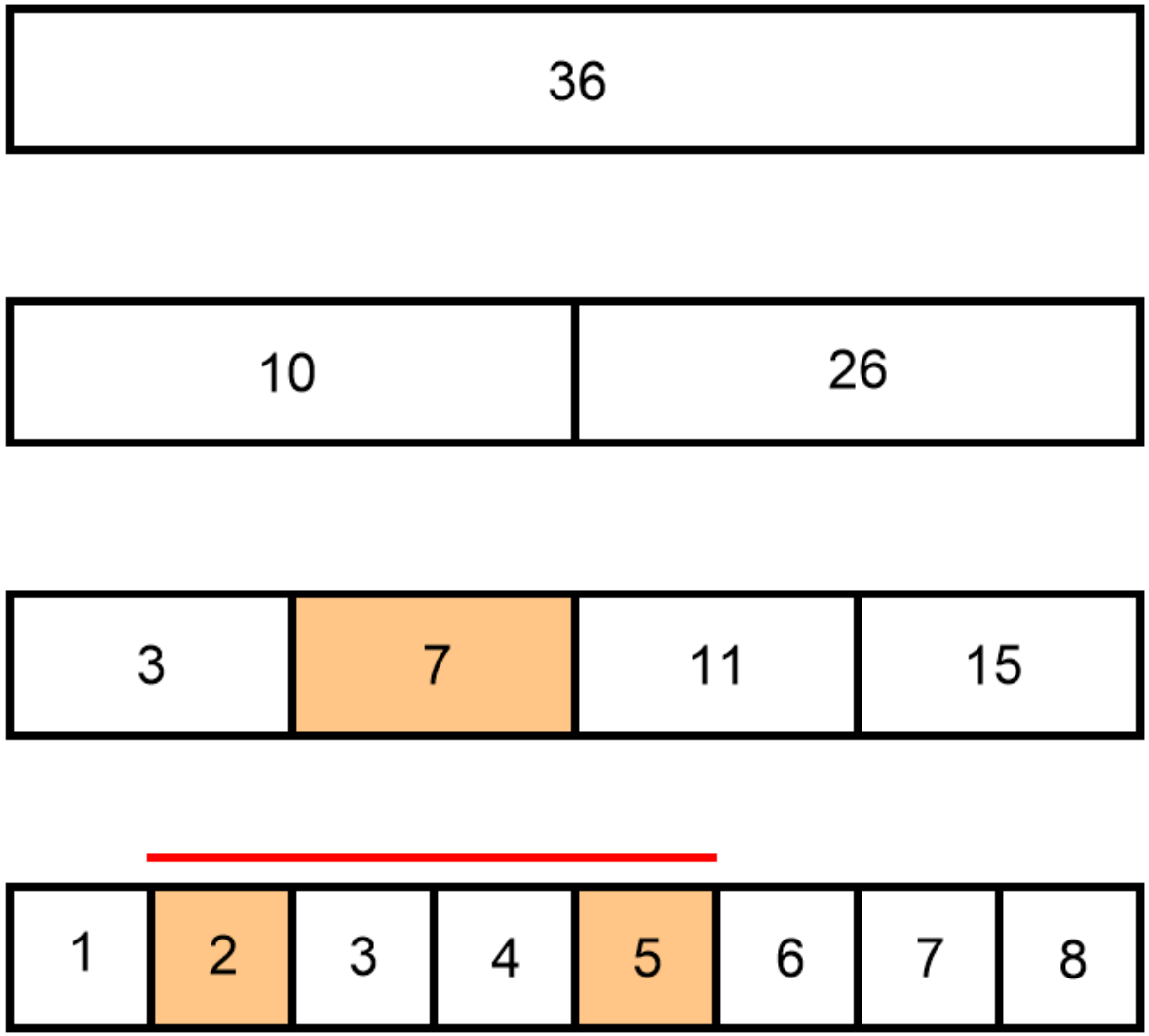

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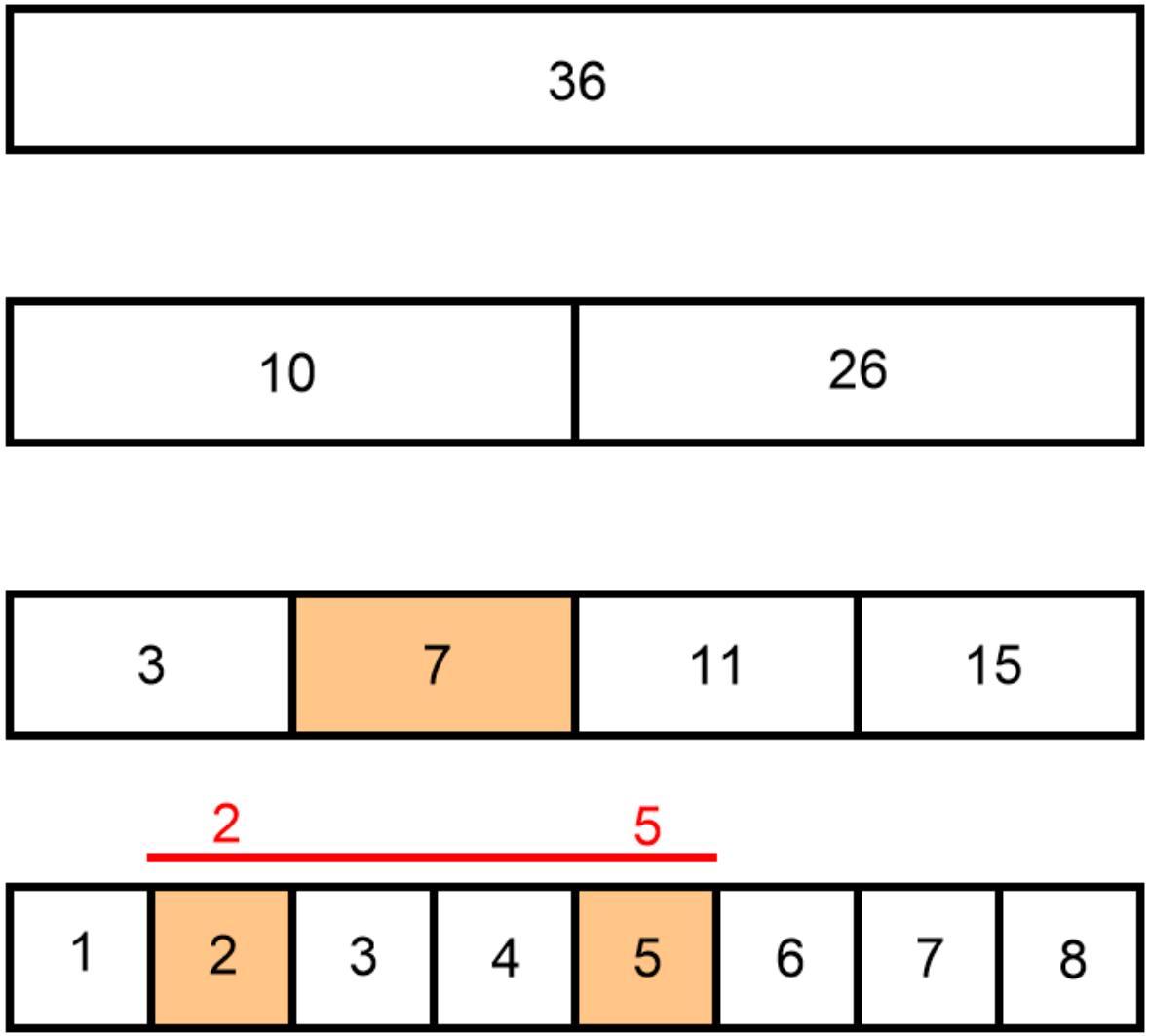

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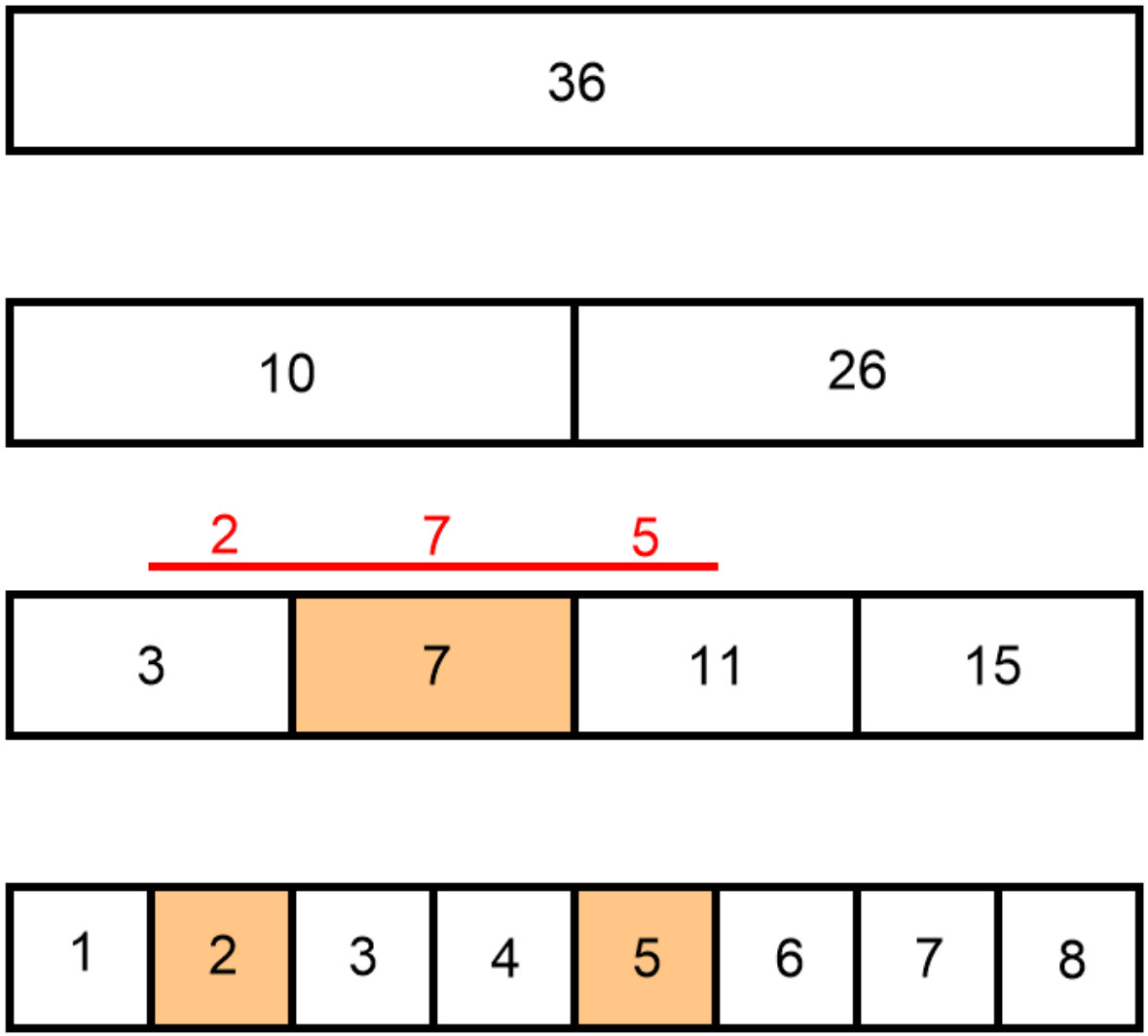

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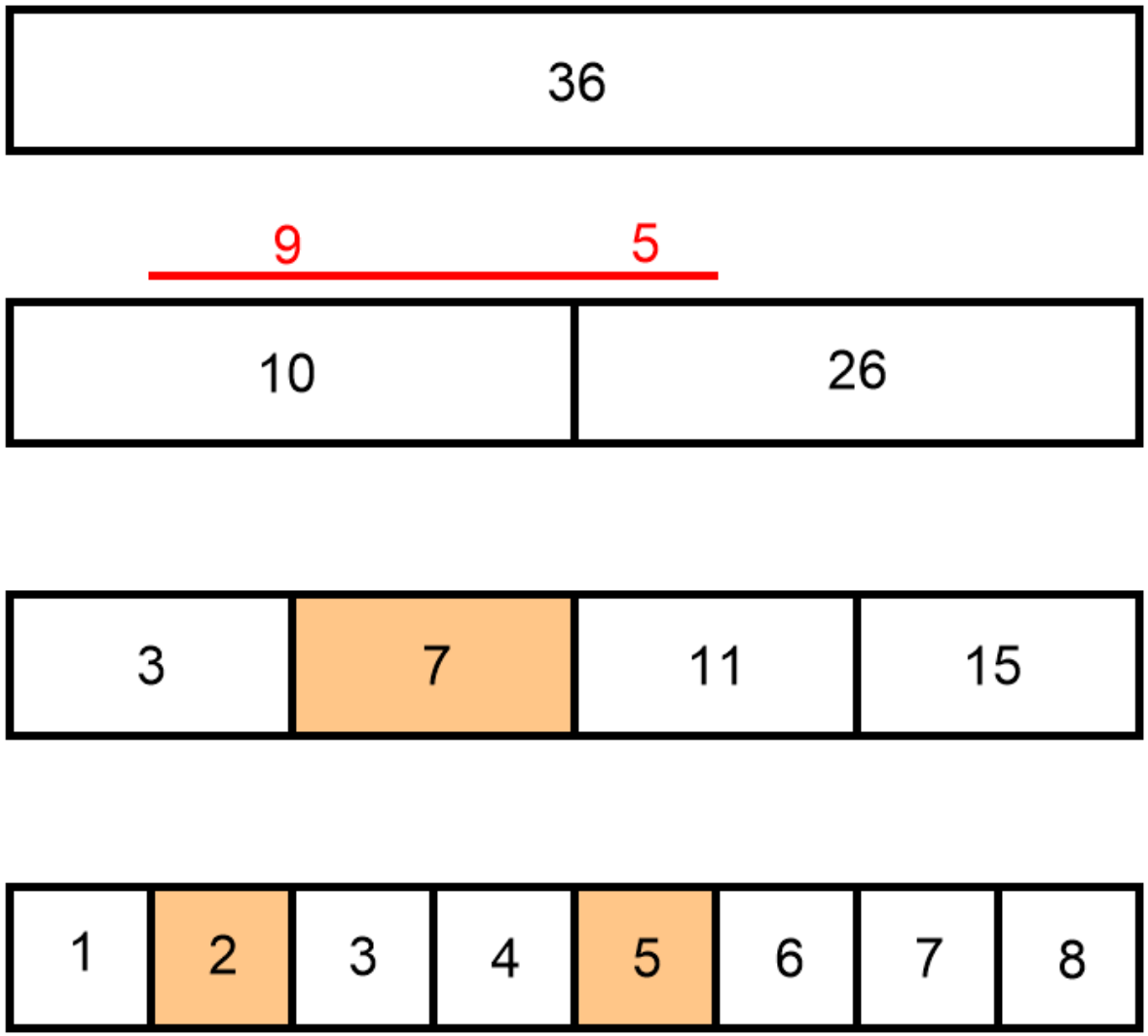

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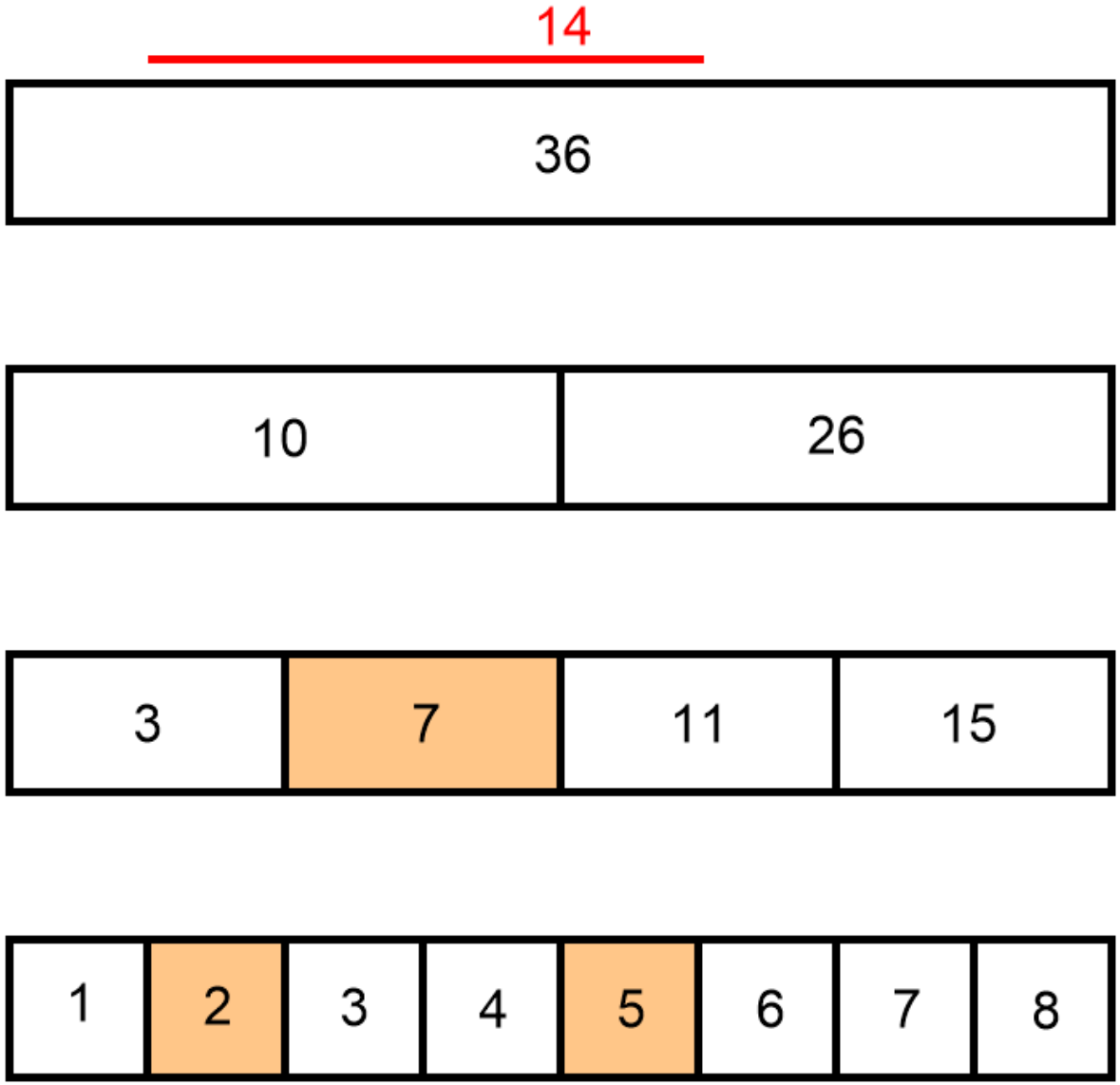

Segment Tree - Construction

- You could call update on each element $\Rightarrow O(n \log n)$
- Better: fill from leaf upwards layer by layer $\Rightarrow O(n)$

Segment Tree - Implementation

Approach is similar to e.g. heap implementation

- Store the tree in an array $A[1 \dots 2n]$.
- Root = $A[1]$.
- Children of node i = $A[2i]$ and $A[2i + 1]$.
- Parent of node i = $i/2$.

Segment Tree - Initialize

```
1  // Set input size as some large power of 2.
2  const int MAXN = 1 << 17;
3
4  // Store segment tree in a flat array.
5  int T[2*MAXN];
```

Segment Tree - Build

```
1  void build(int A[MAXN]) {
2    // initialize leaf
3    for (int i = 0; i < MAXN; i++)
4      T[MAXN + i] = A[i];
5
6    // initialize internal nodes, bottom up
7    for (int i = MAXN-1; i > 0; i--) {
8      T[i] = T[2*i] + T[2*i+1];
9    }
10 }
```

Segment Tree - Point Update

```
1  void update(int x, int val) {
2    // Change the leaf node's value.
3    int v = MAXN + x;
4    T[v] = val;
5
6    // Propagate the change all the way to the root.
7    for (int i = v/2; i > 0; i /= 2)
8      T[i] = T[2*i] + T[2*i + 1];
9  }
```

Segment Tree - Range Query

```
1  int query(int x, int y, int i=1, int l=0, int r=MAXN-1)
2    // [l,r] is completely outside [x,y], return 0
3    if (x > r || y < l) return 0;
4
5    // [l,r] is completely in [x,y], return node value
6    if (x <= l && r <= y) return T[i];
7
8    // Otherwise, recurse on children.
9    return query(x, y, 2*i, l, (l+r)/2)
10        + query(x, y, 2*i+1, (l+r)/2+1, r);
11 }
```

Problem 1

Support the following operations on an array $A[1 \ldots n]$

- *update* (same as before)
- *query(l, r)* returns the maximum sum subarray within this subrange.

Problem 1 - Solution

Consider a range $[l, r]$. How could we get the maximum sum subarray in this range?

- The answer lies entirely in the left child or entirely in the right.
- The answer could spans the left and right children.
- So $ans(l, r) = \max\{ans(left), ans(right), sum(across)\}$.

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How do we get the maximum sum subarray that goes across the left and right children?

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How do we get the maximum sum subarray that goes across the left and right children?

Clearly, we need to store more information in the nodes.

Problem 1 - Solution

Store these information in each node

- Sum of entire subrange contained in the node = *node.sum*
- Best prefix sum = *node.prefix*
- Best suffix sum = *node.suffix*
- Answer = *node.ans*

Update queries

- *node.sum = left.sum + right.sum*
- *node.prefix = max(left.prefix, left.sum + right.prefix)*
- *node.suffix = max(right.suffix, left.suffix + right.sum)*
- *node.ans = max(left.ans, right.ans, left.suffix + right.prefix*

Time Complexity: $O(\log n)$.

Range Updates

So far we have only been dealing with “point updates”, where we change only a single input element at a time.

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Suppose now we also need to be update an interval $[l, r]$ – for example, add 3 to every element in an interval. How do we do this?

Range Updates

So far we have only been dealing with “point updates”, where we change only a single input element at a time.

Suppose now we also need to be update an interval $[l, r]$ – for example, add 3 to every element in an interval. How do we do this?

Naive way: call *update* once for each element in the interval. Time Complexity: $O(n \log n)$! Can we do better?

Lazy Propagation

Basic idea

- From range query: range $[l, r]$ = union of ranges represented by $O(\log n)$ nodes, so we can just update value at these nodes!
- Idea: at these nodes, add a "todo" variable saying every leaf in the subtree needs to be updated by x, don't recurse further
- Update ancestors as usual

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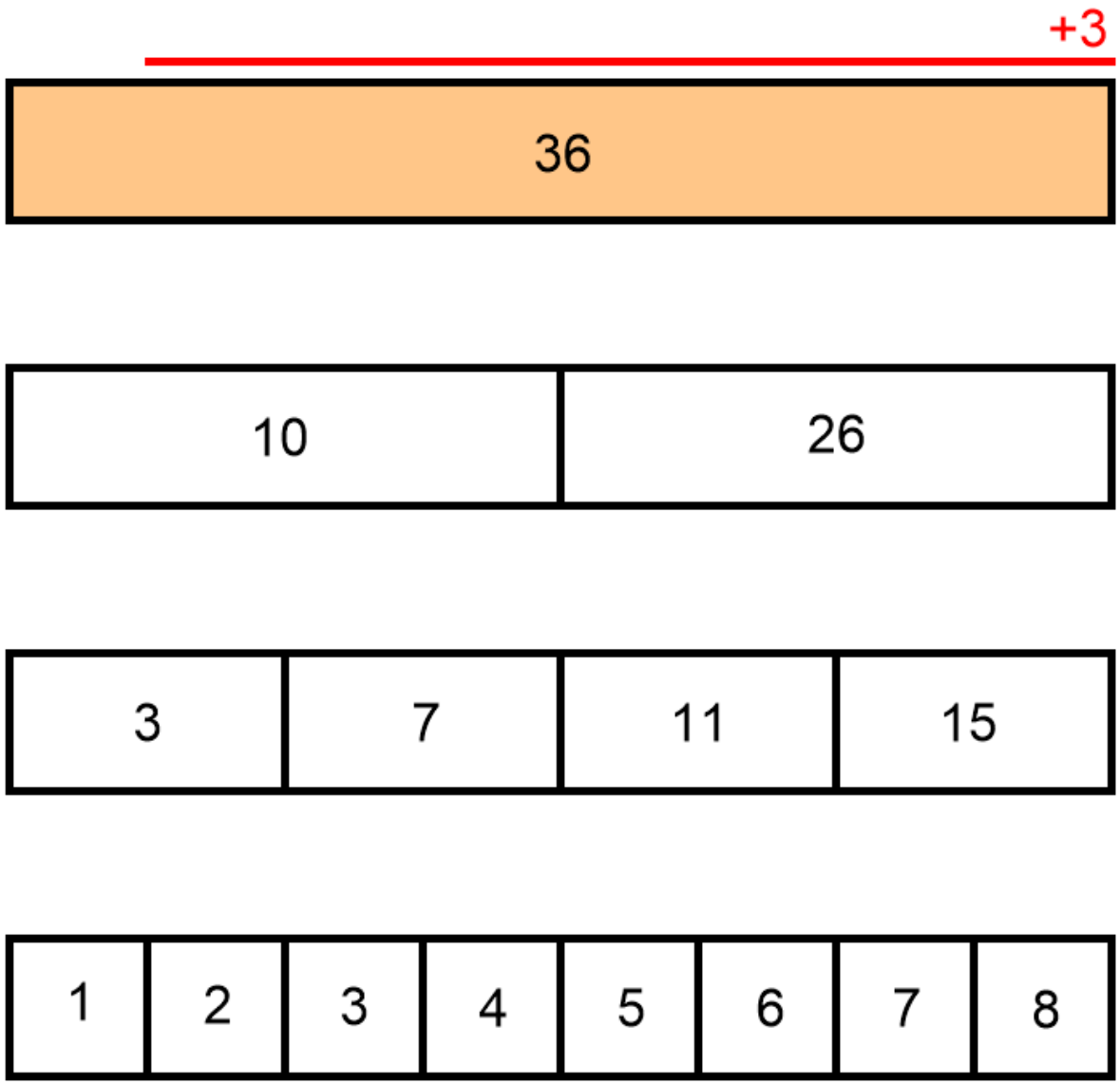
- Idea: at these nodes, add a "todo" variable saying every leaf in the subtree needs to be updated by x, don't recurse further
- Update ancestors as usual

What if end point of next update/query interval lands in the middle of a node with a "todo"?

- Before recursing on children, push the "todo" down one level
- After recursing on children, continue as usual

Lazy Propagation – Lazy Update



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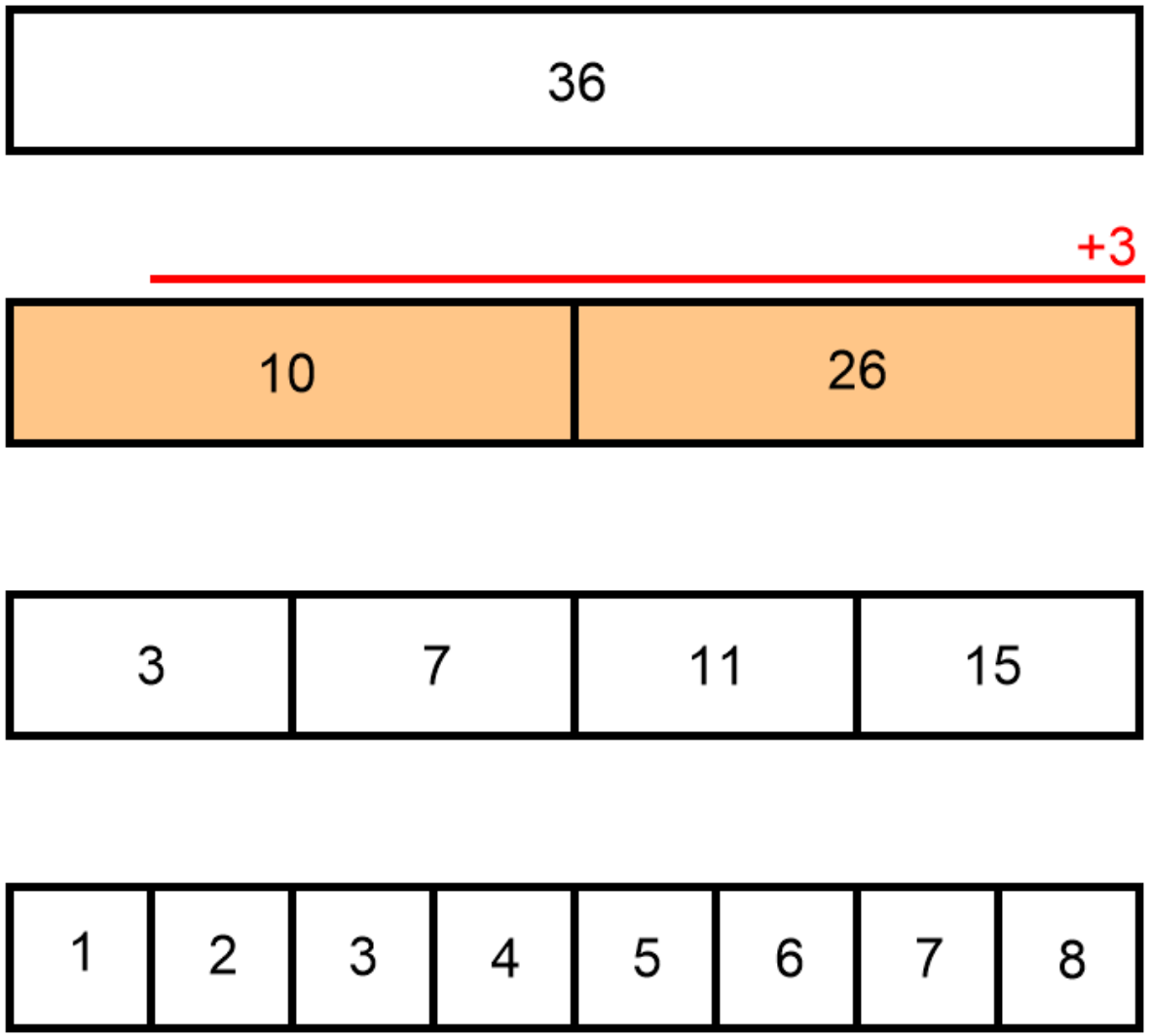

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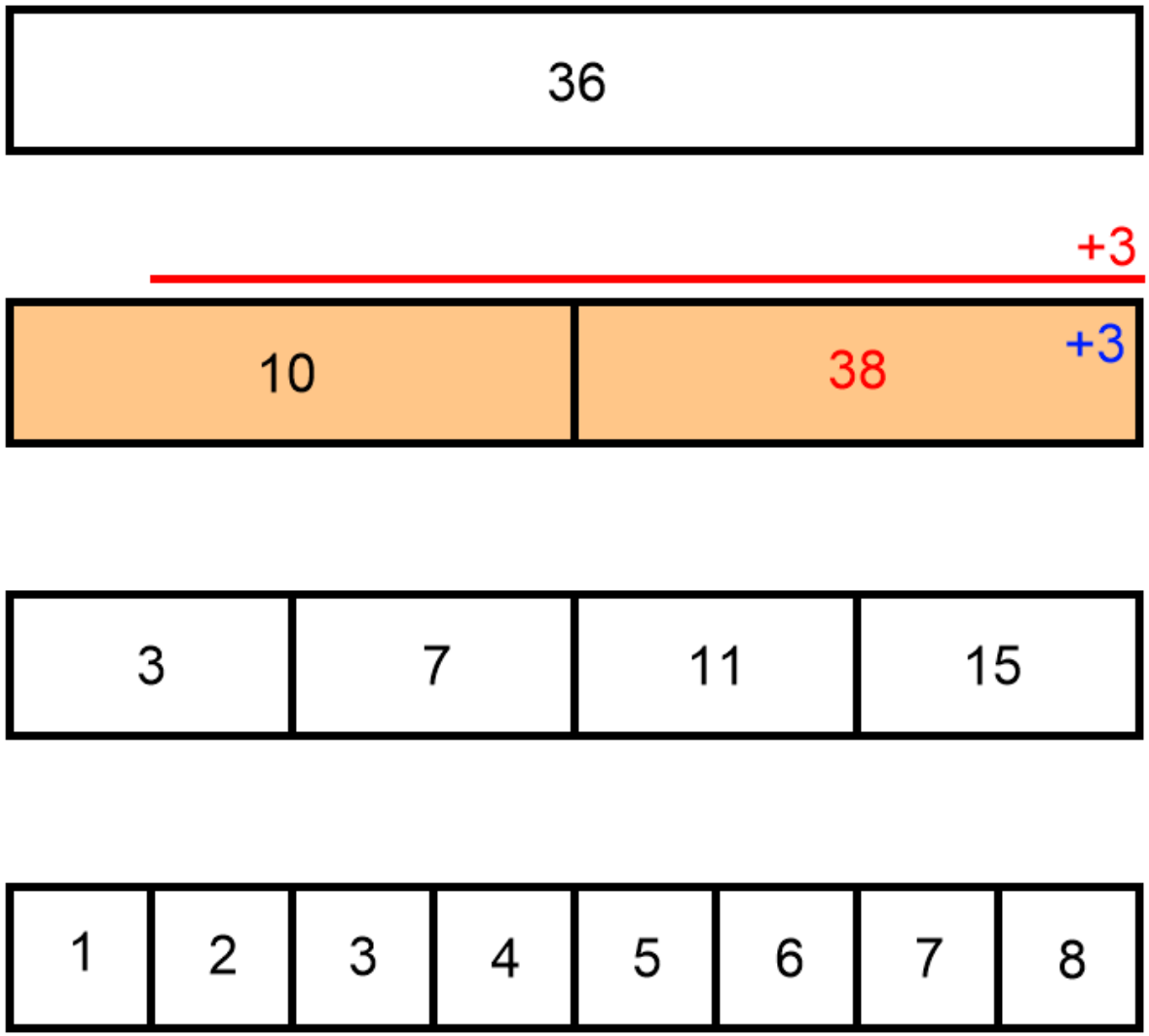

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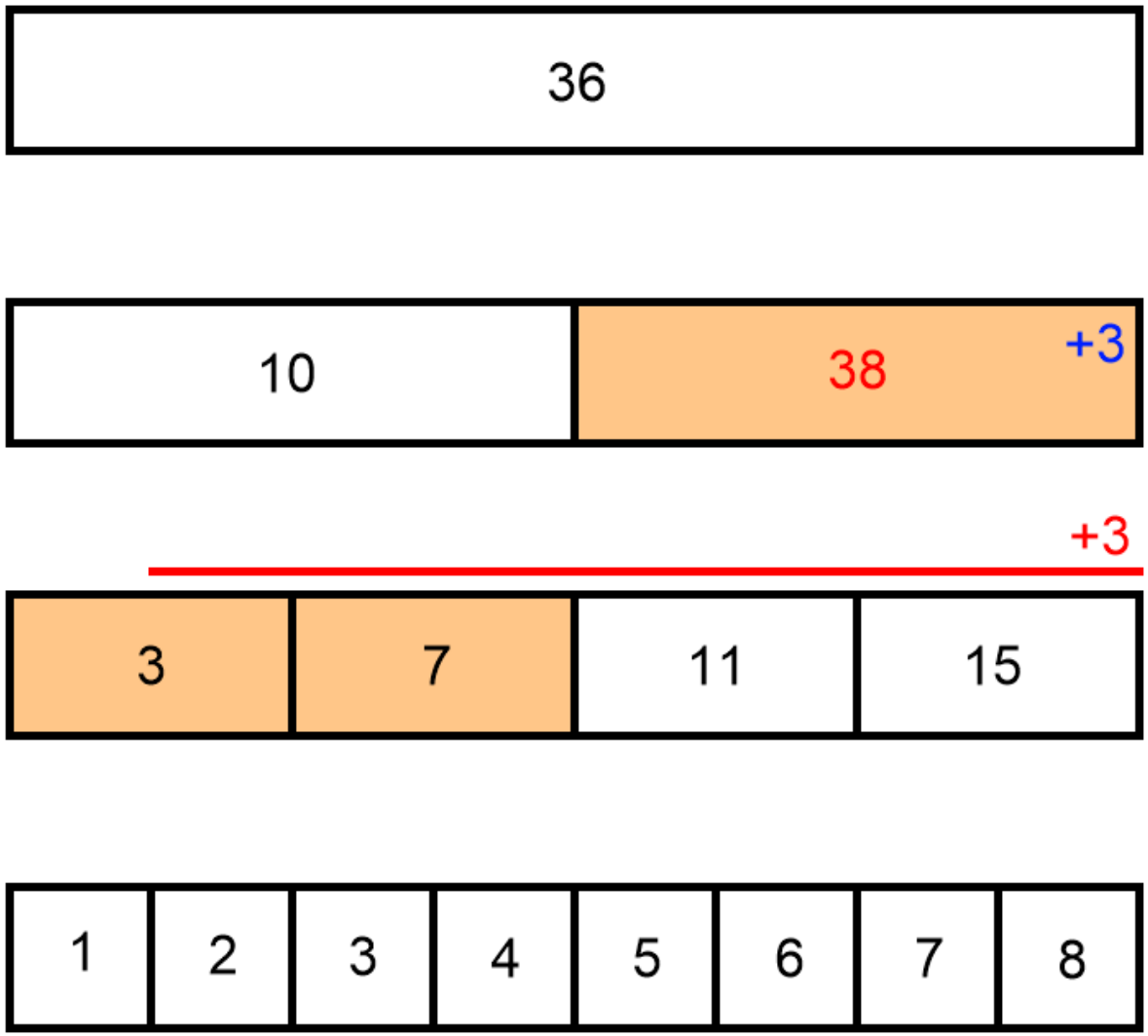

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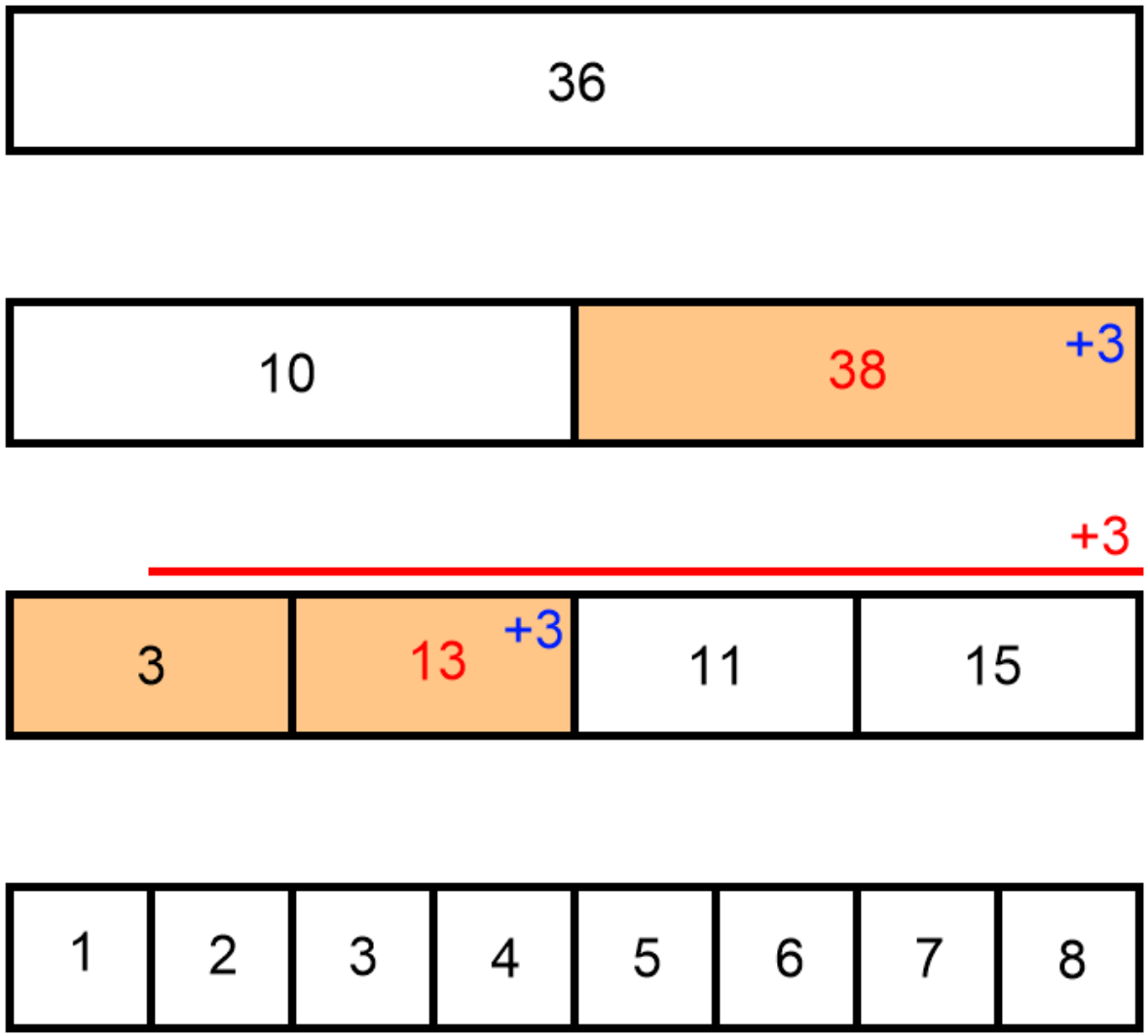

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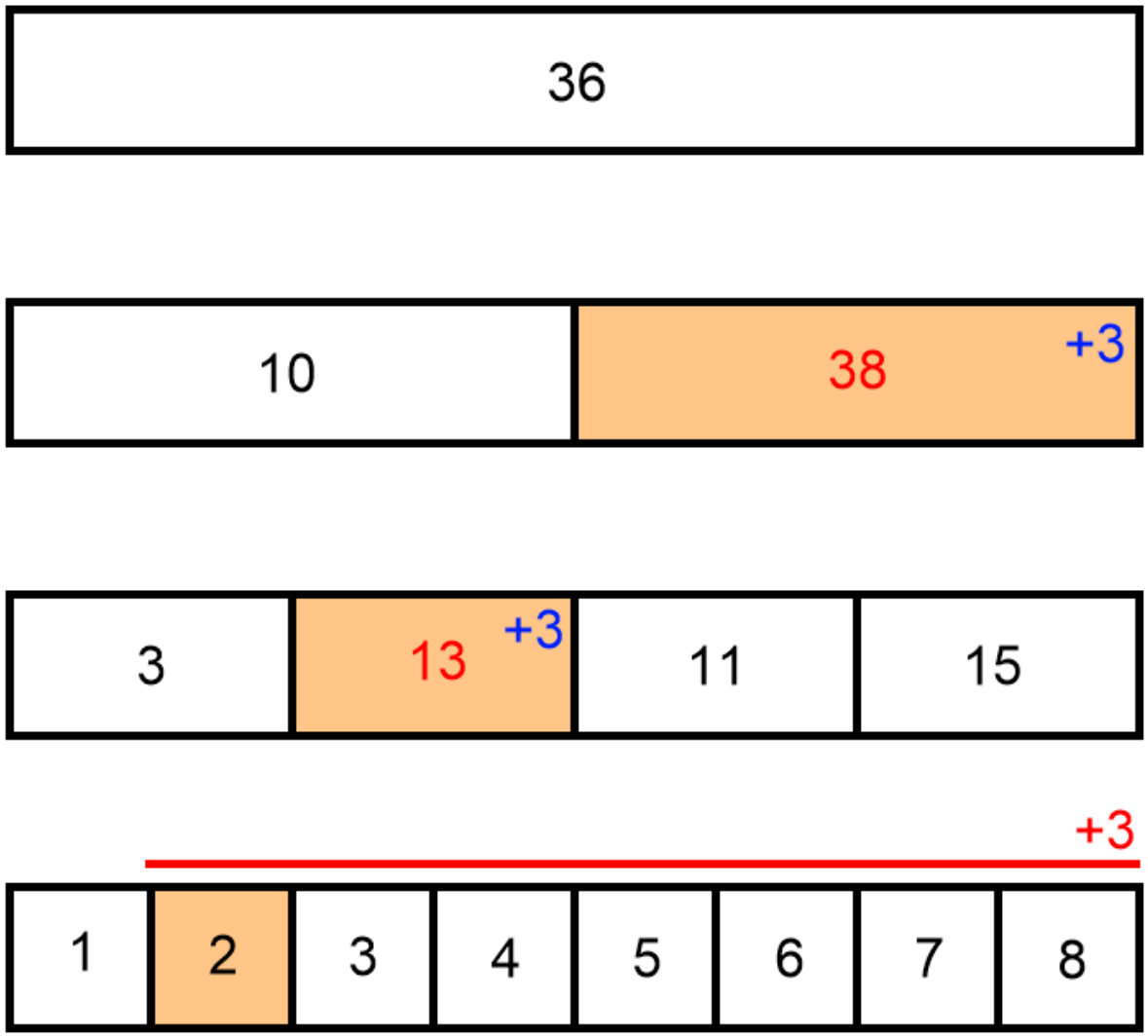

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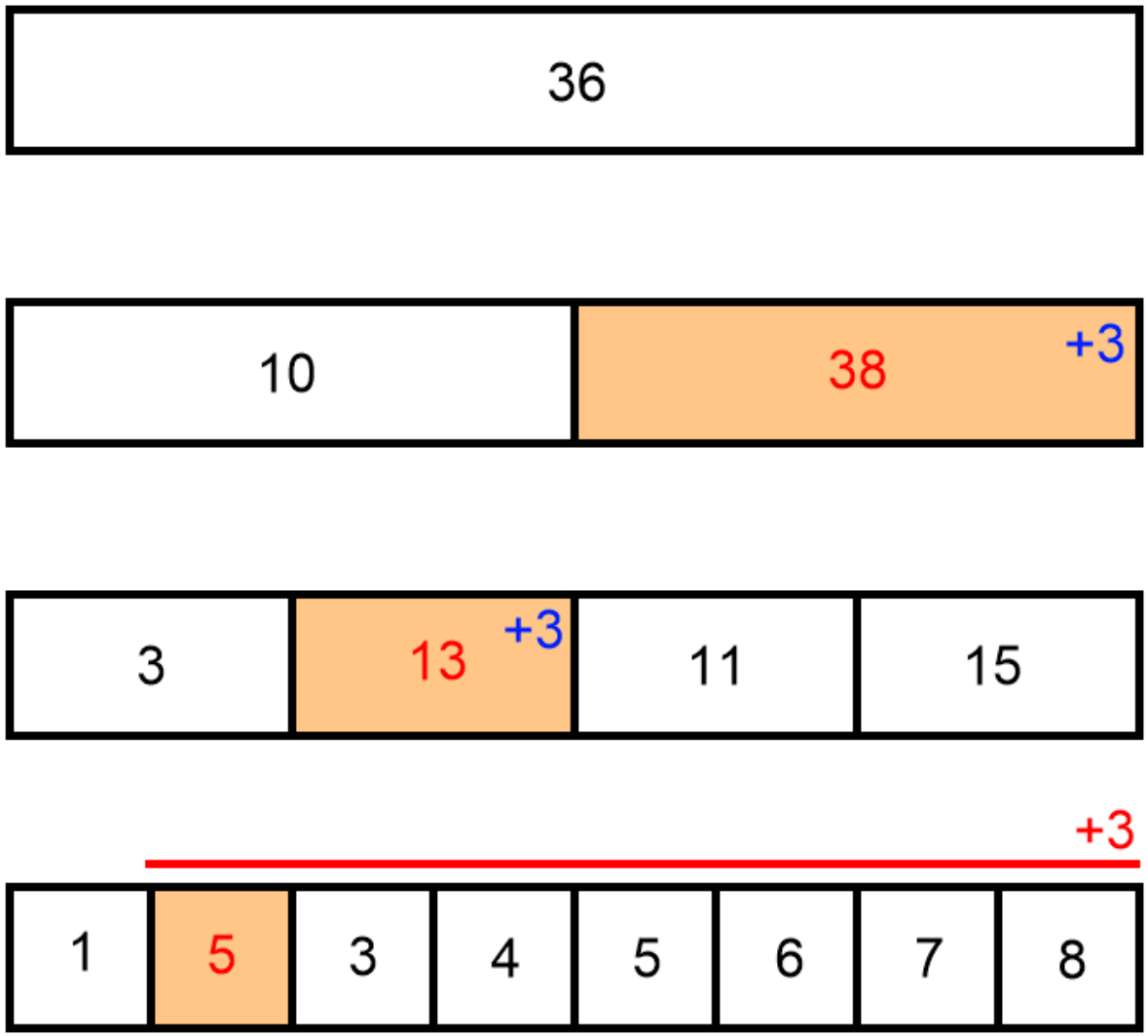

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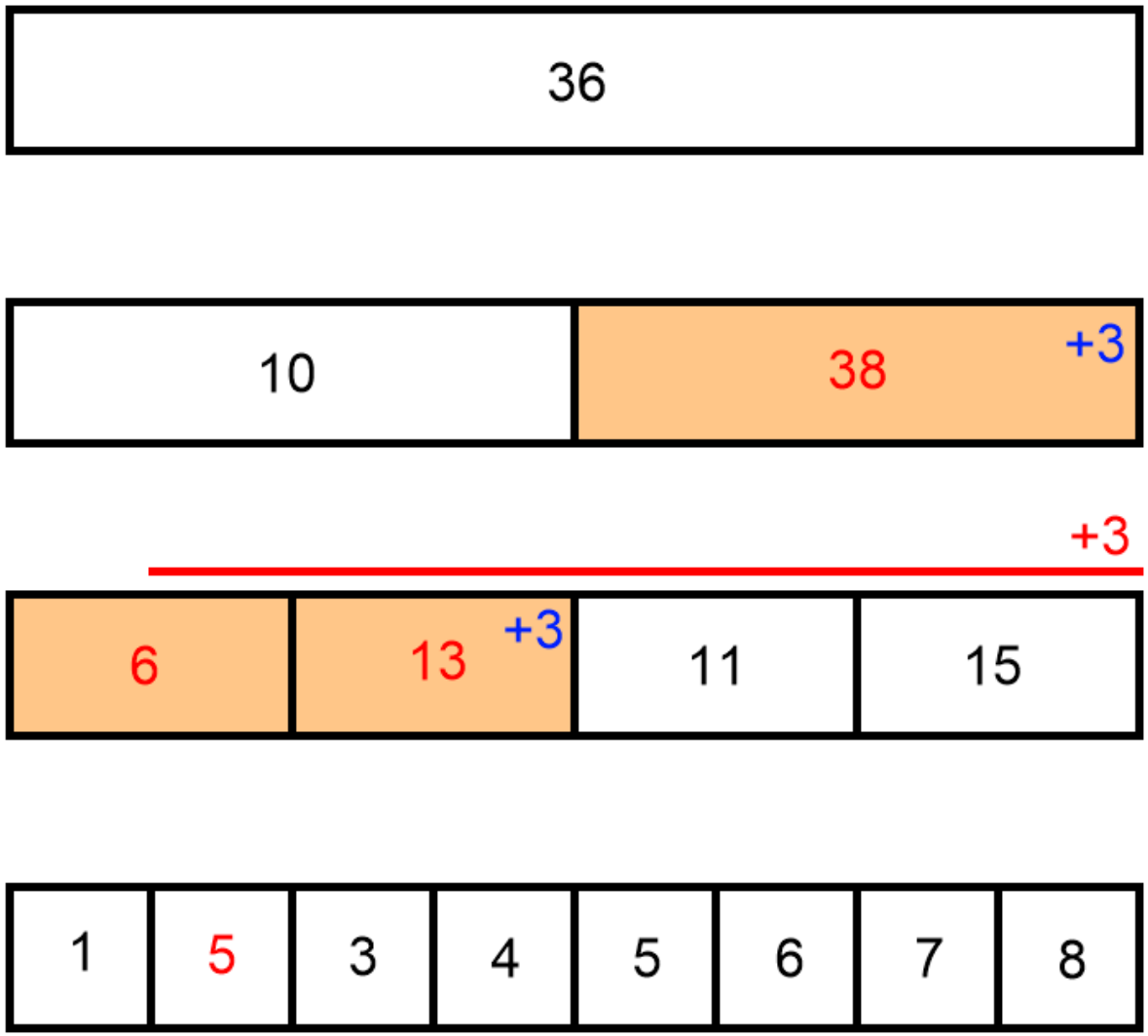

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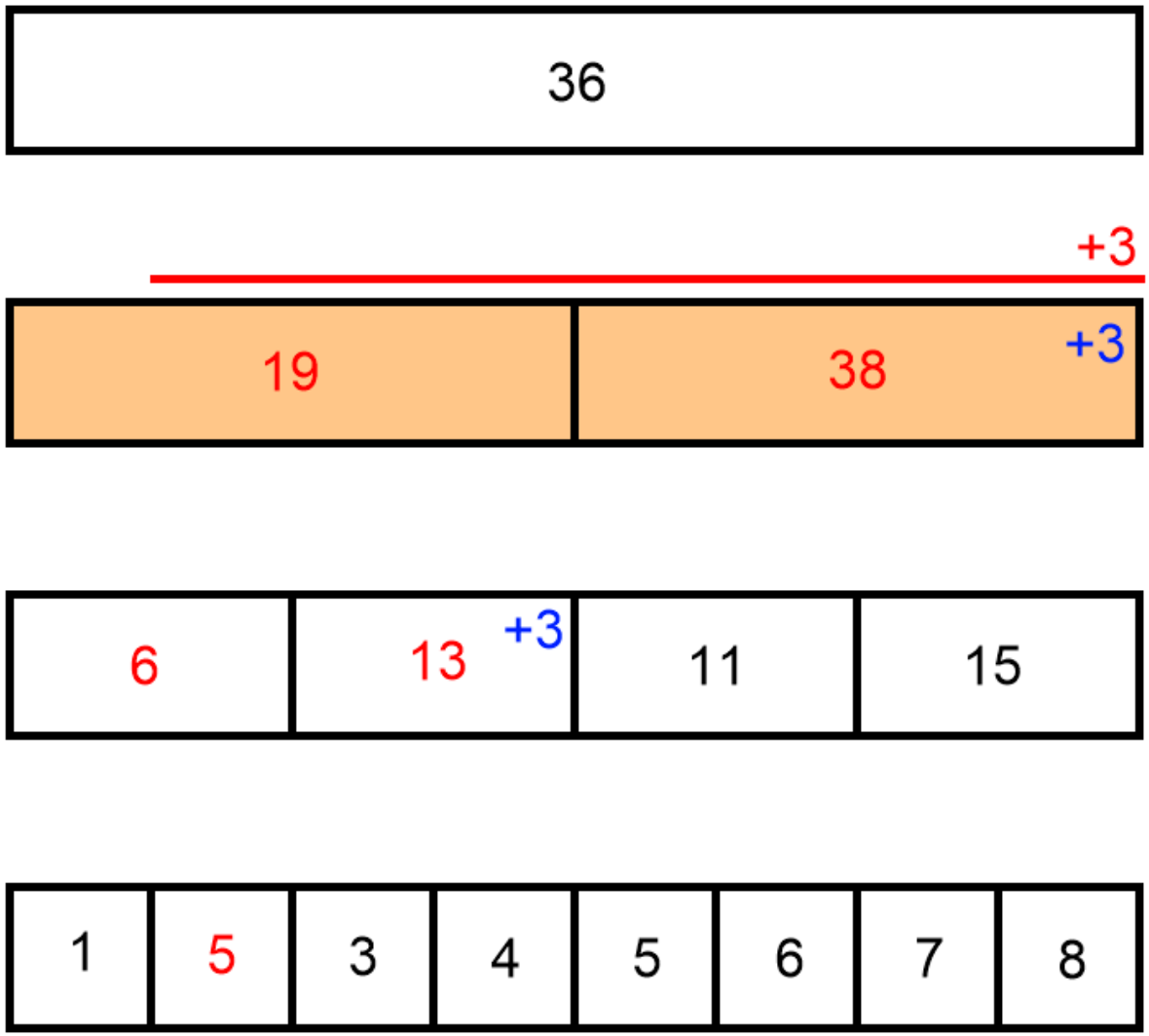

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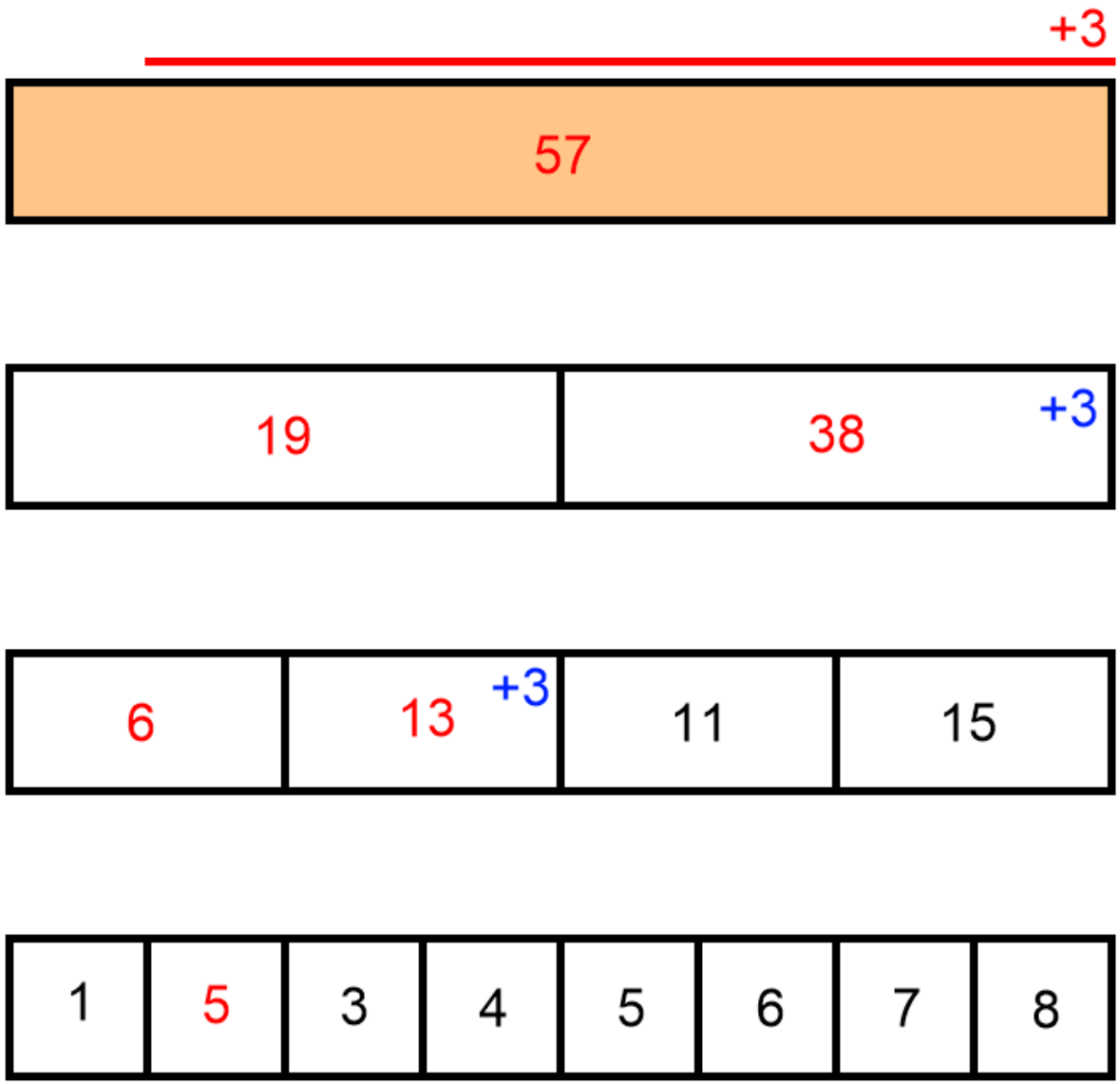
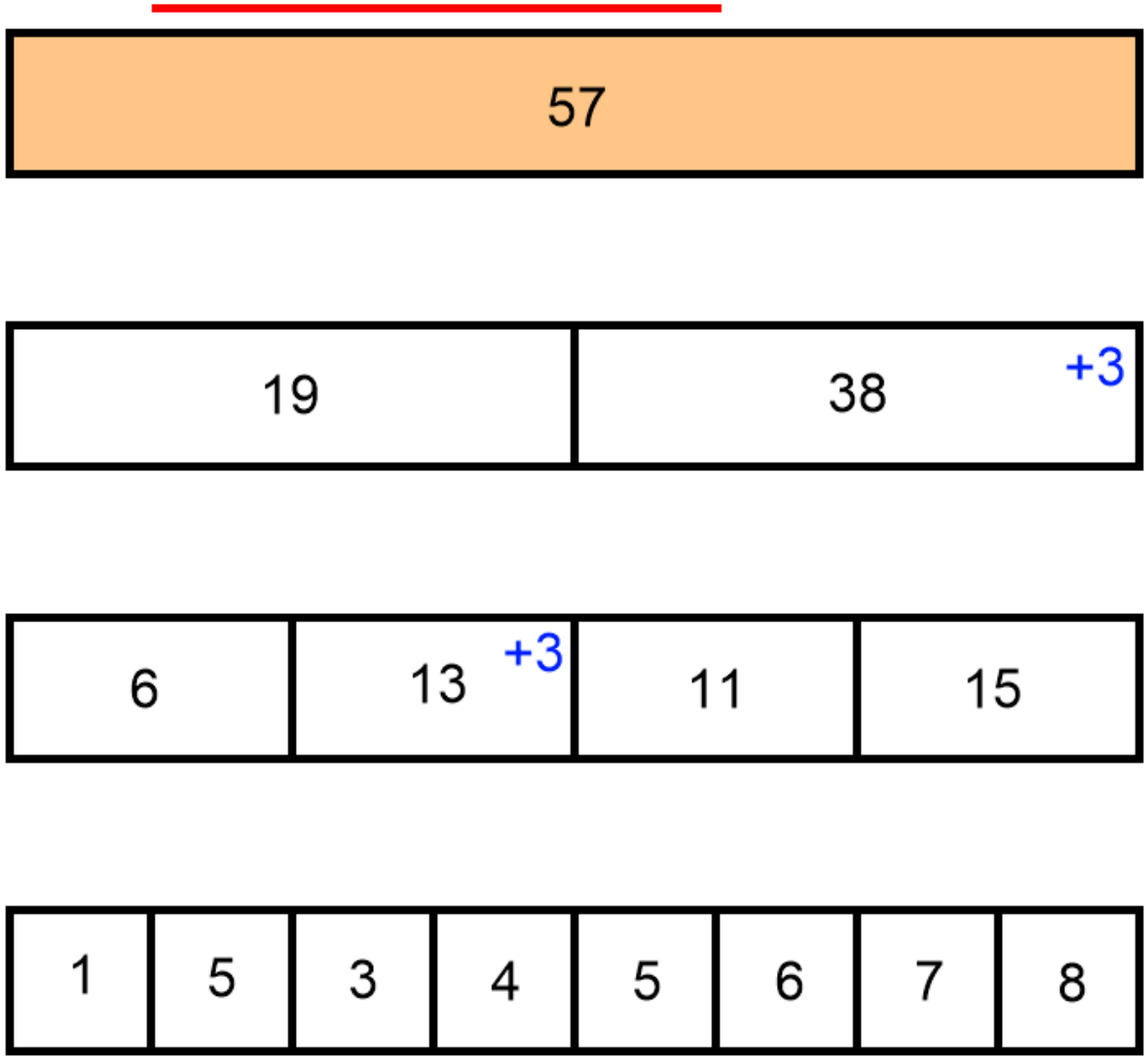


Lazy Propagation – Range Query



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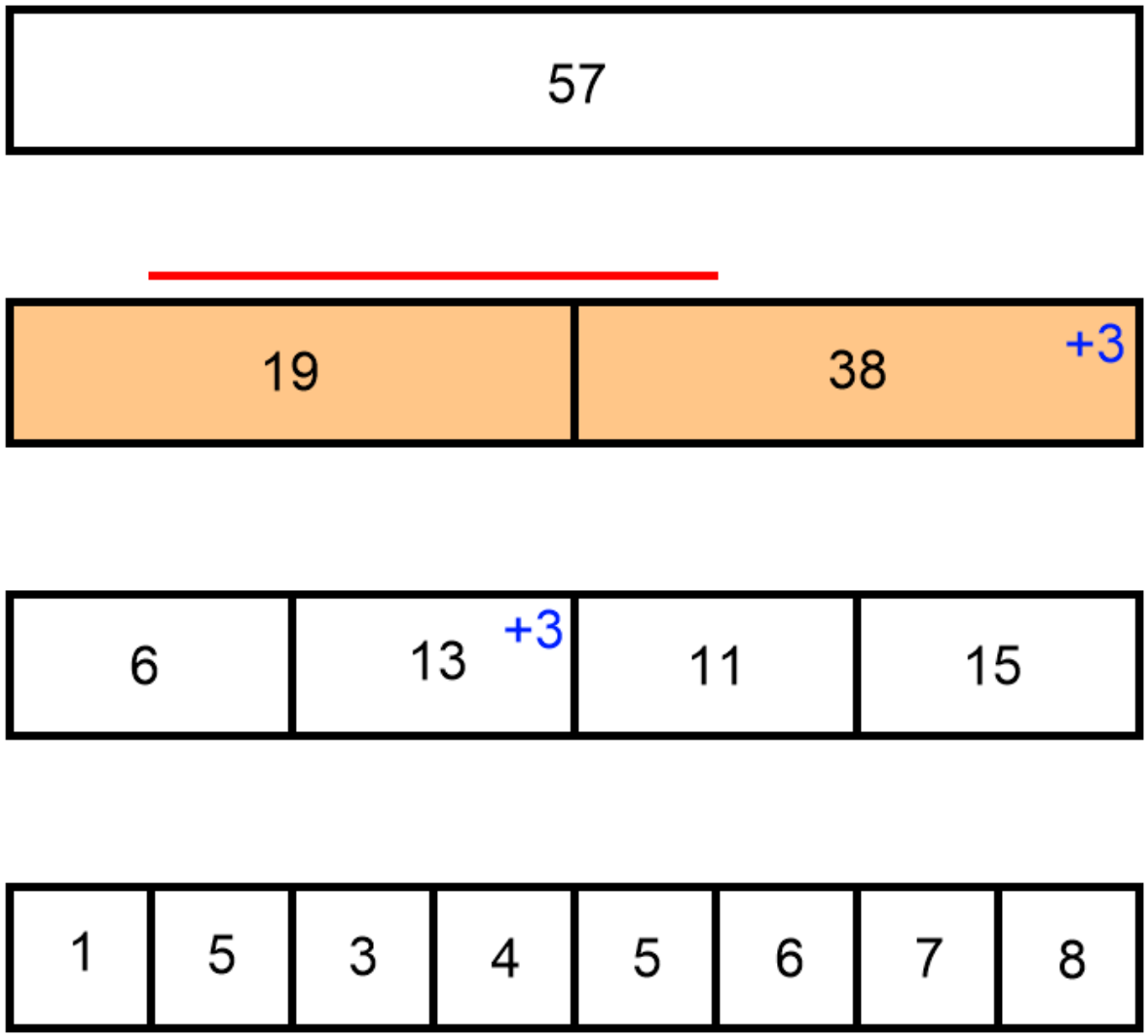

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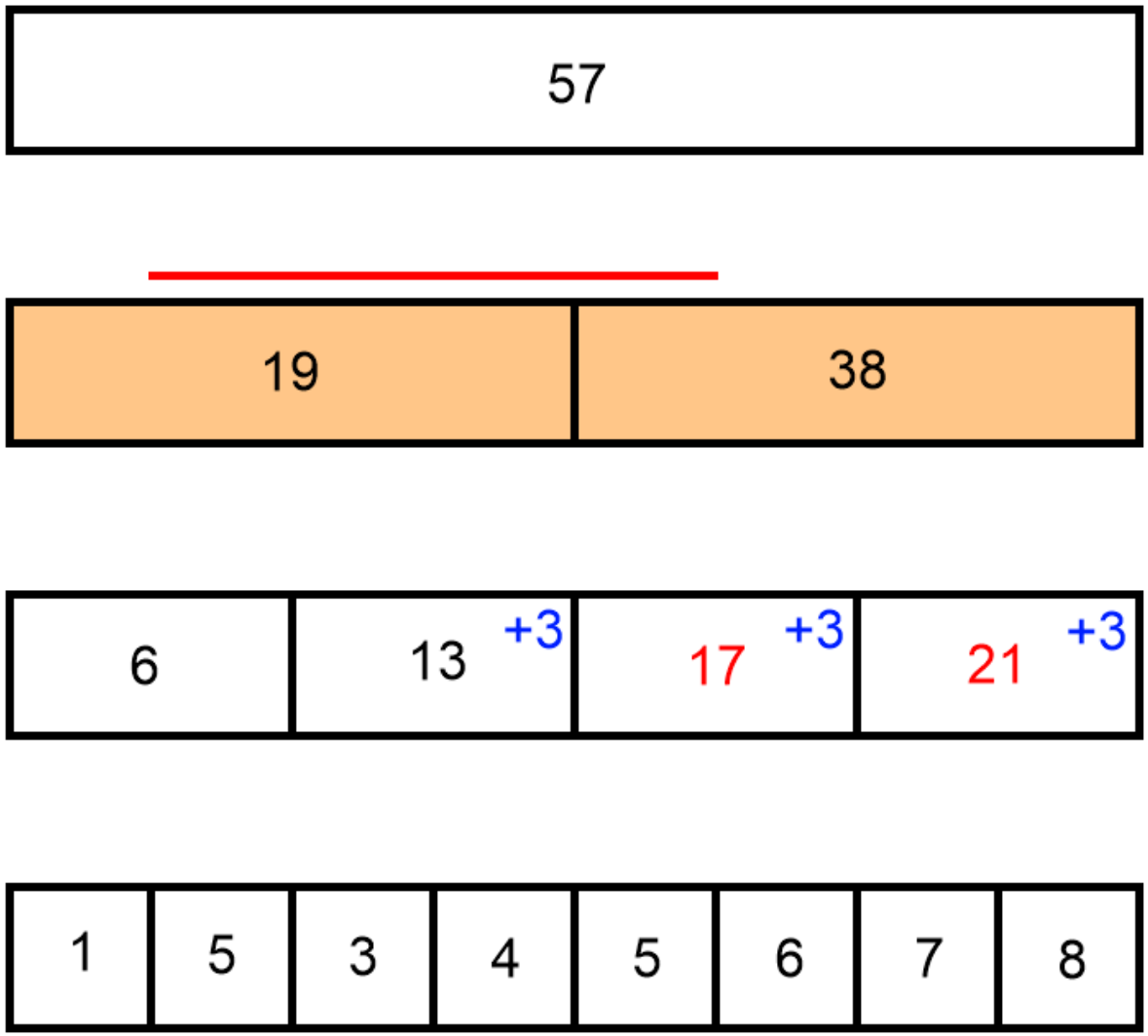

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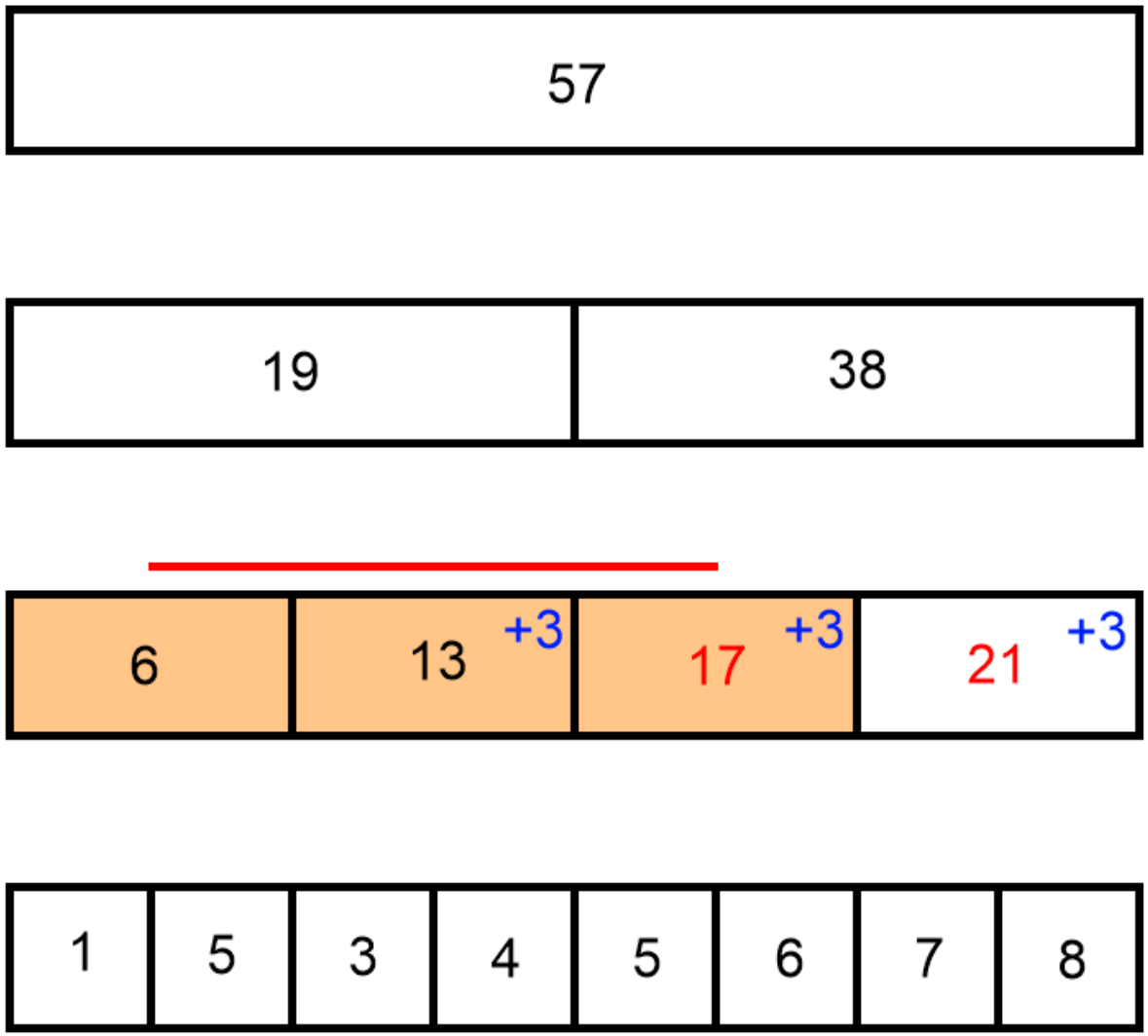

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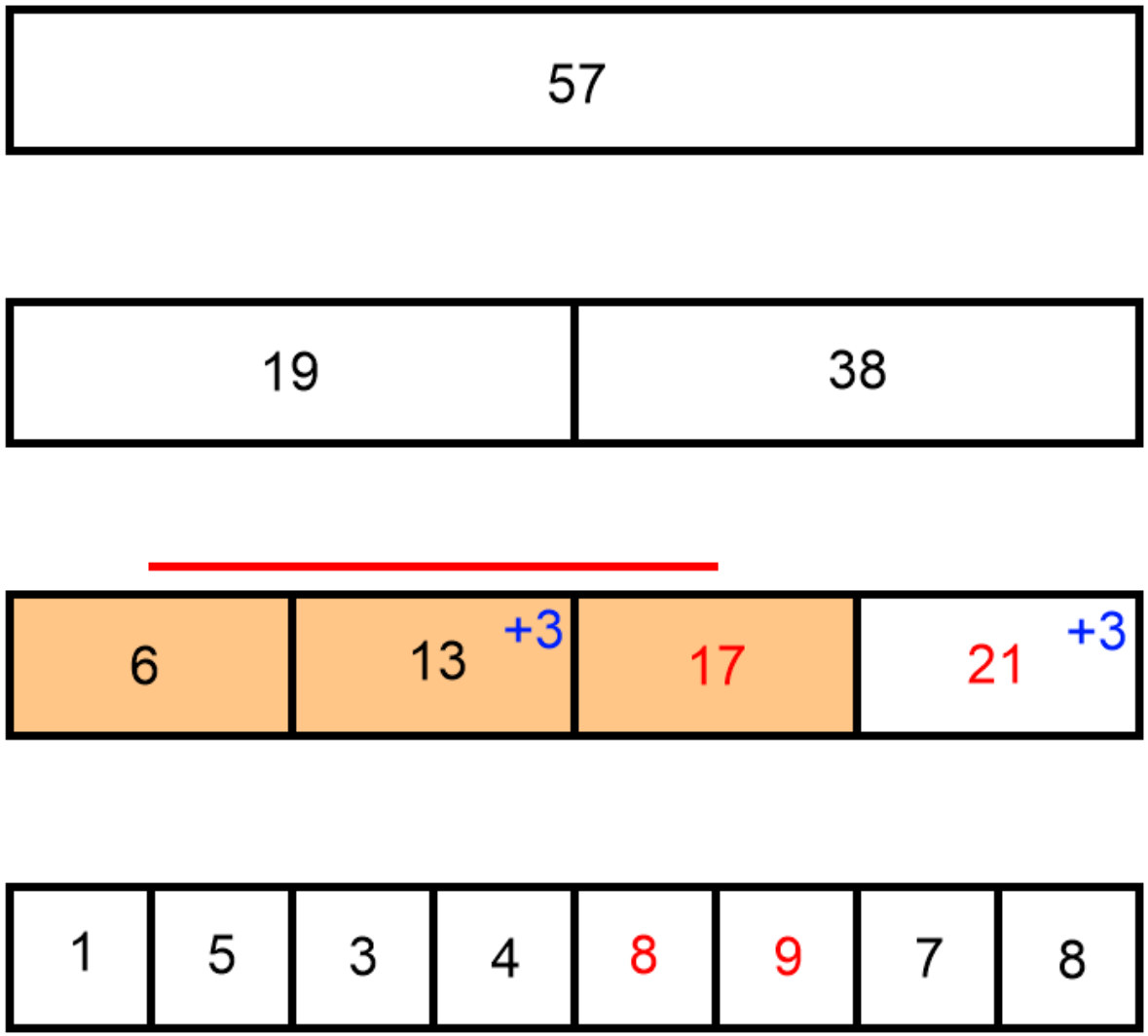

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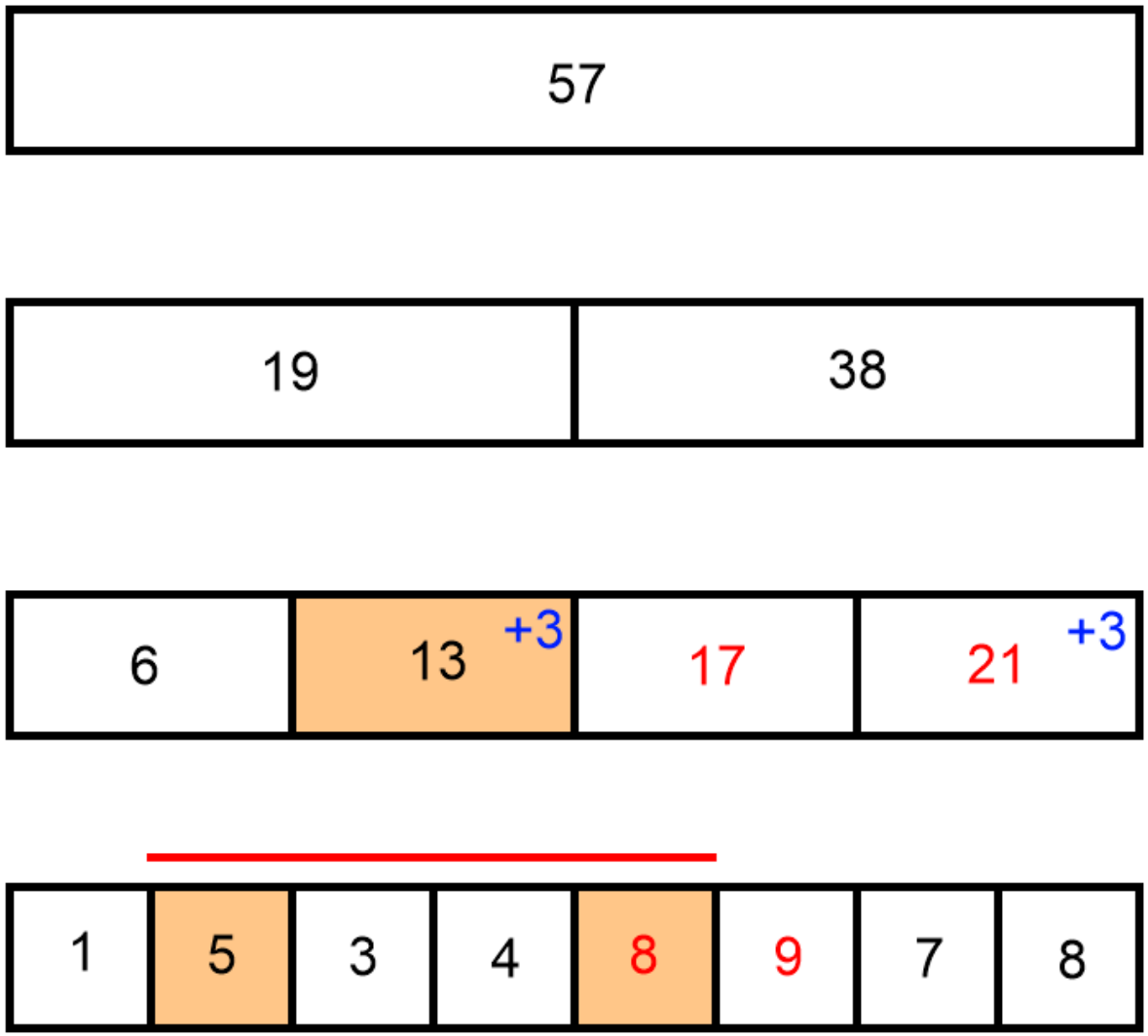

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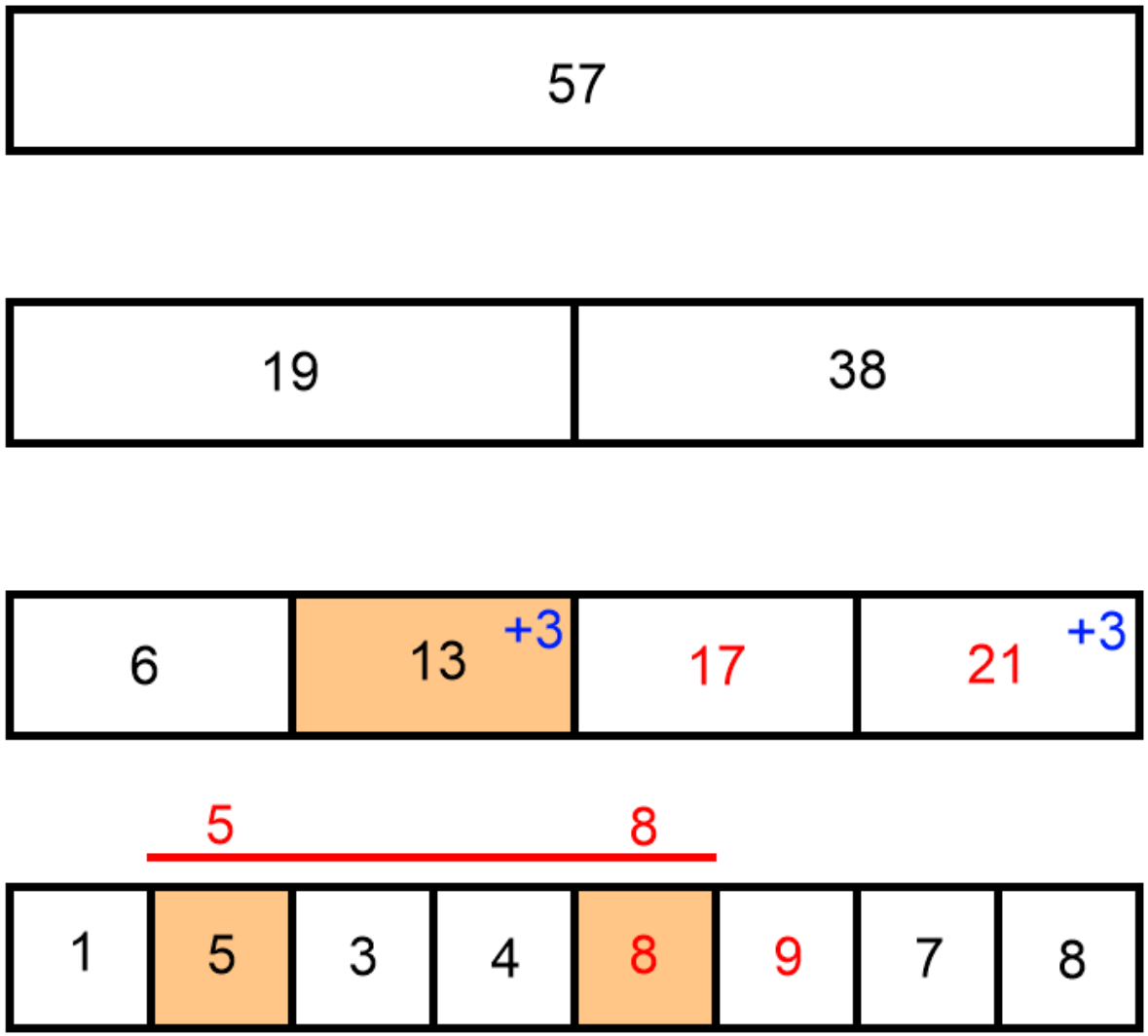

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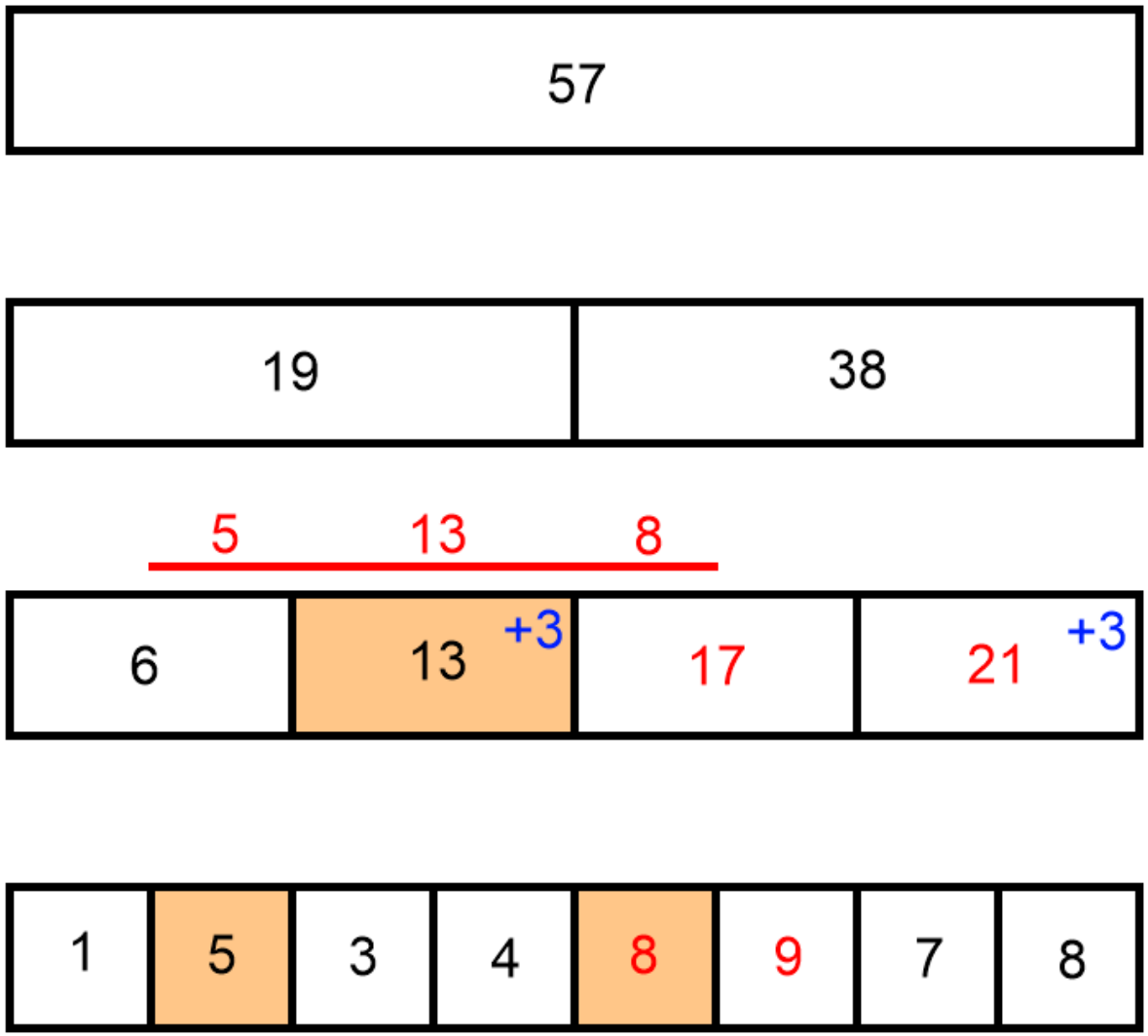

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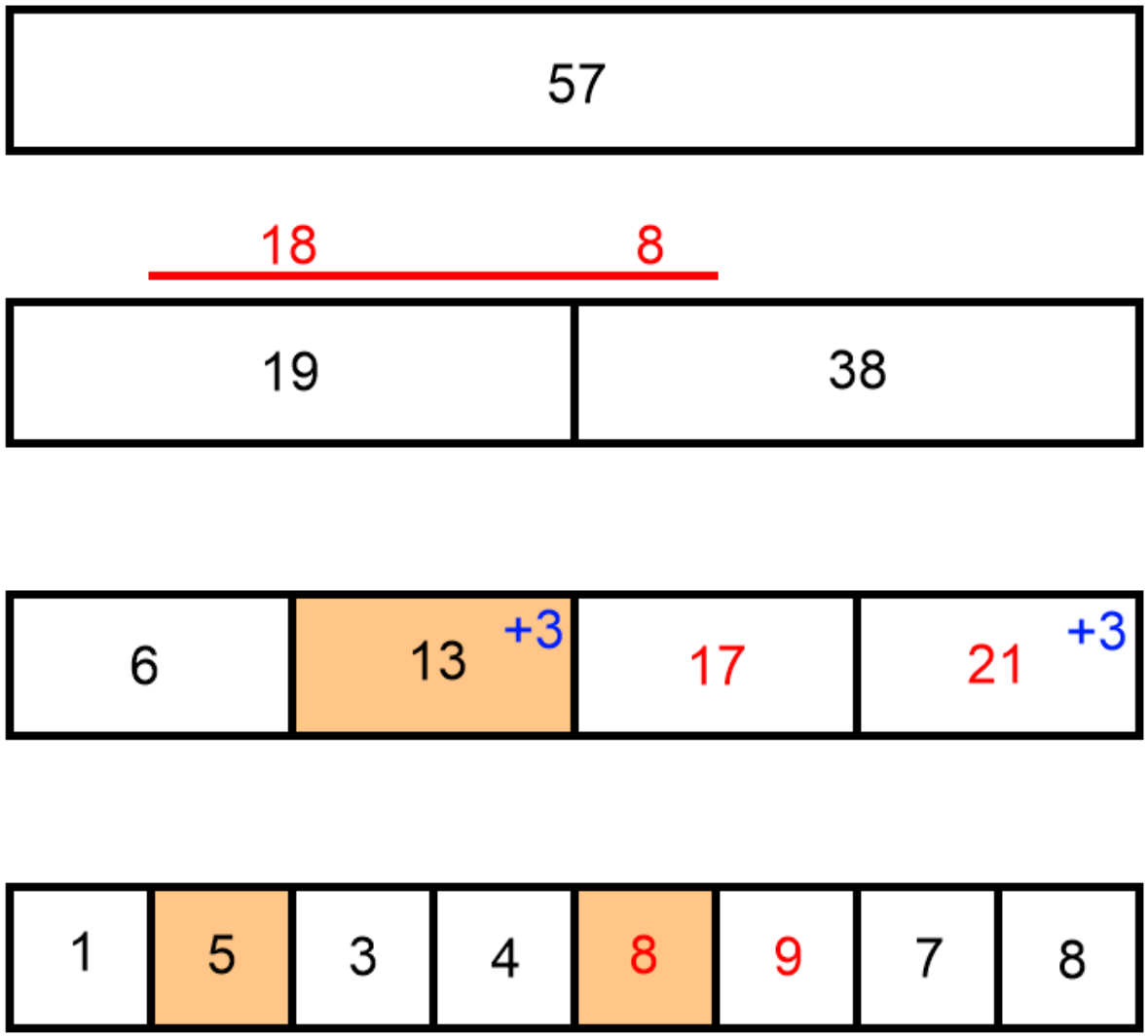

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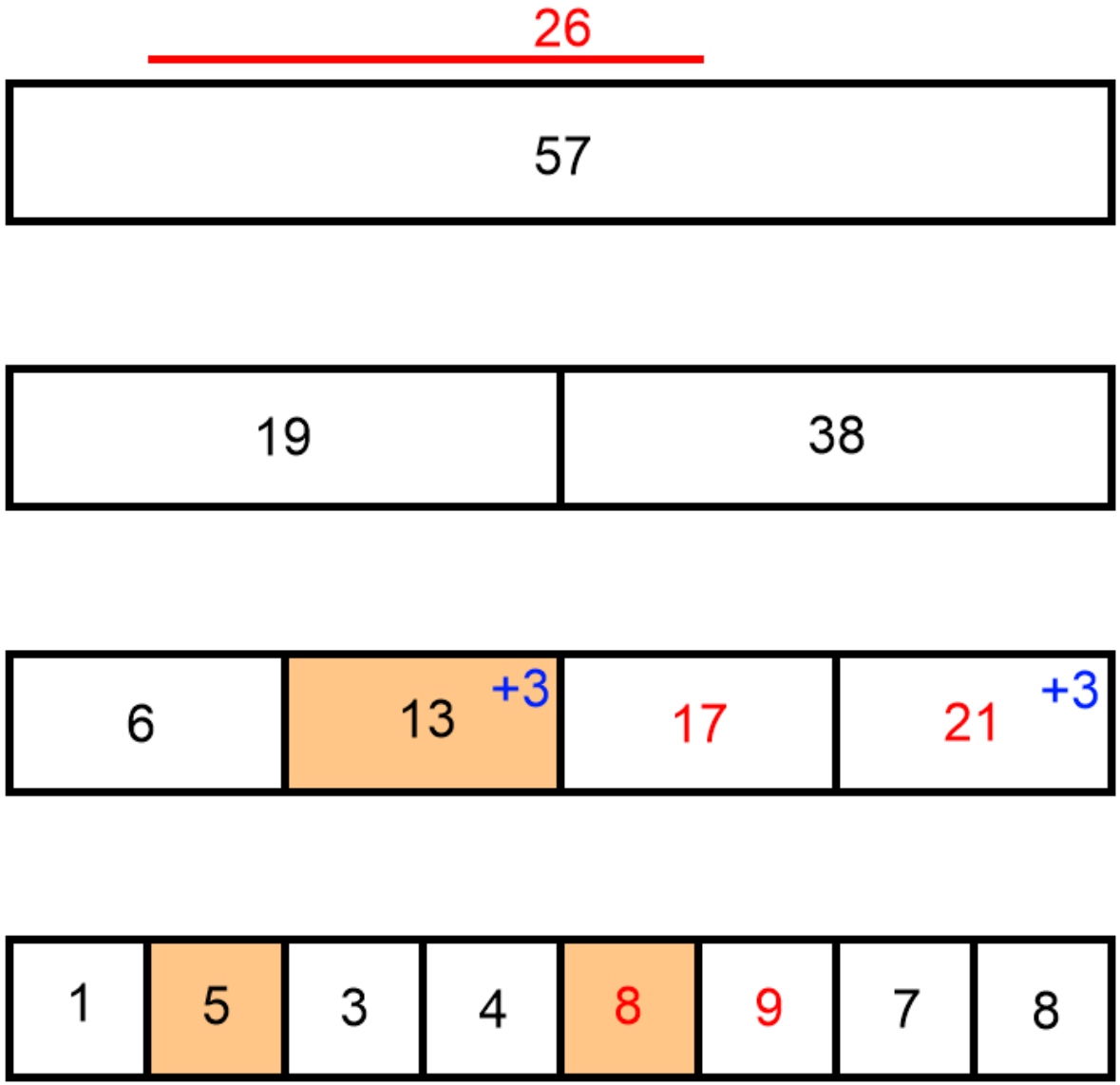

Next Class

Binary Indexed Tree,
other tree tricks