CPSC 490 – Problem Solving in Computer Science

Lecture 9: KMP, Trie

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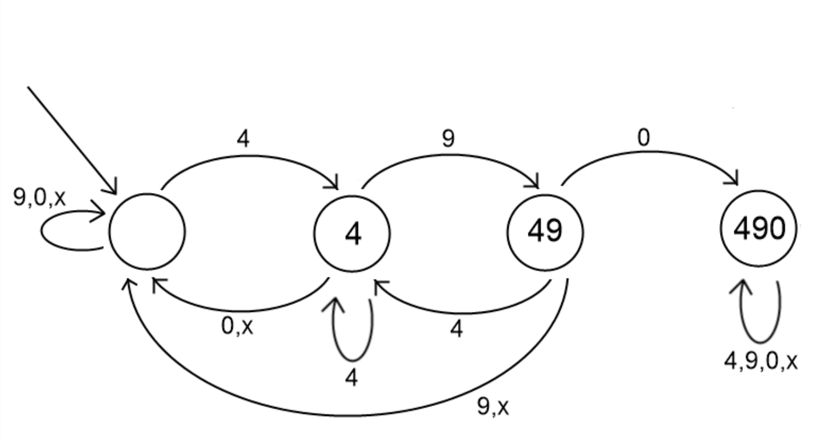
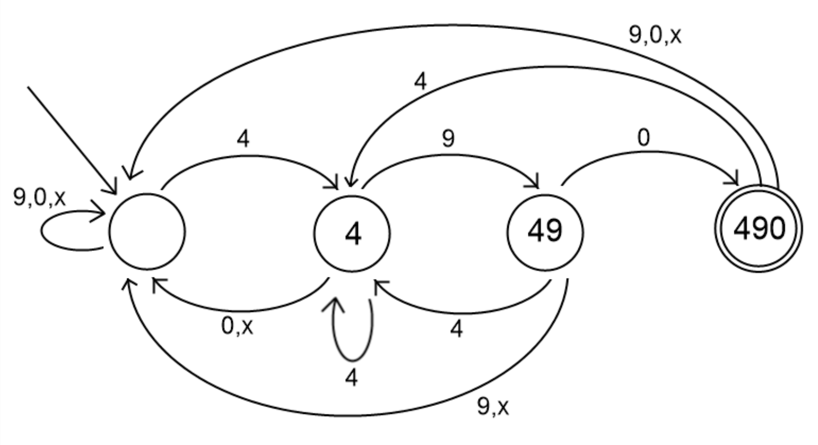
University of British Columbia

Remember this DFA?



What does this slightly different DFA do?



A new algorithm for string matching

We have discovered a super fast algorithm for string matching!

Suppose we want to find search for all copies of string B in string A

- Somehow construct the DFA for string B quickly
- Run string A through the DFA and output (index - len(B) + 1) for all indices when the DFA is in the "completely matched" state

Can we really do this? What's the problem?

Simplifying the DFA

Problem

- Size of the DFA proportional to size of alphabet – could be as bad as the size of the search string ($O(m^2)$, bad!)
- The problem is too many transitions

But we can fix this problem!

Simplifying the DFA

Idea: only need two transitions – success and failure!

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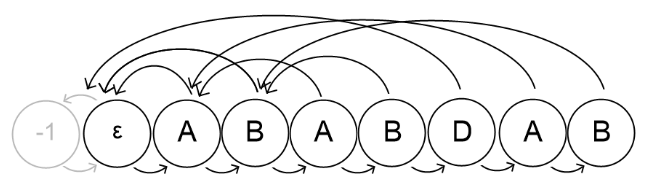

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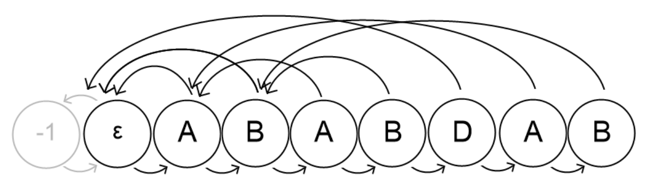

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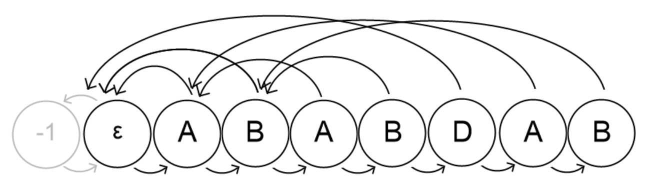

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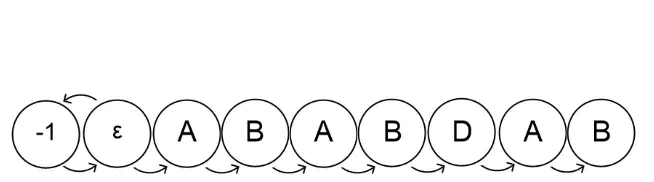

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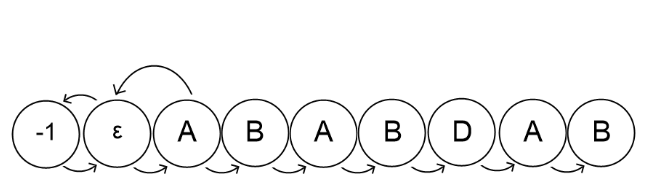

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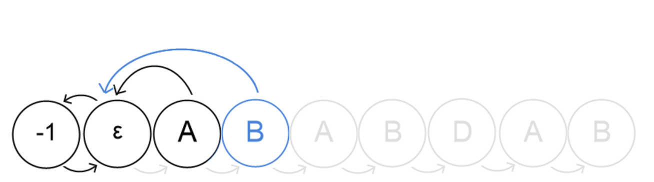

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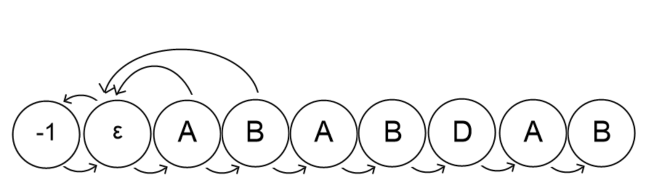

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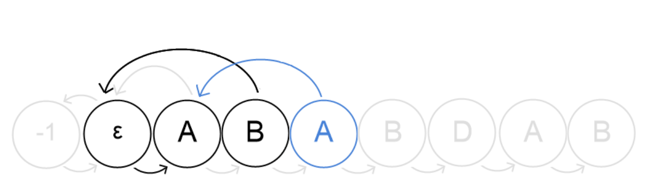

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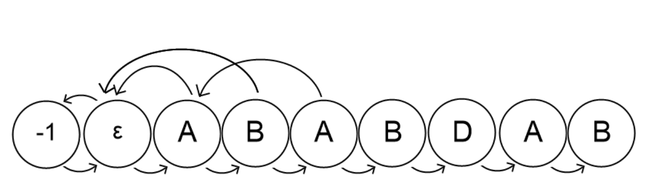

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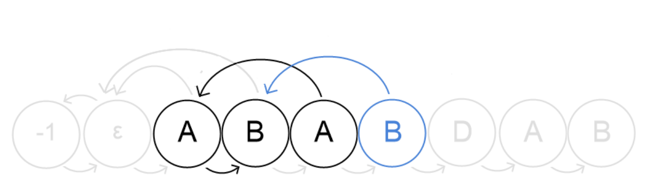

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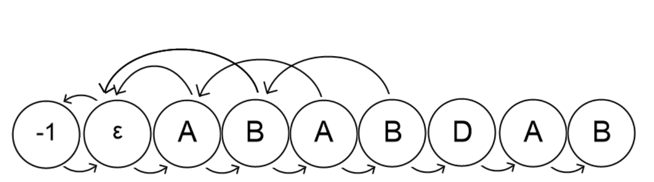

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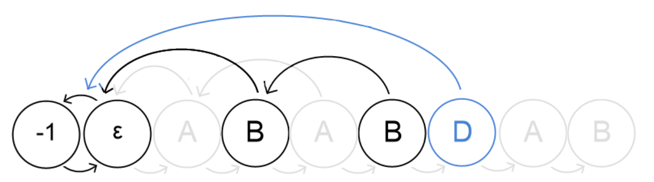

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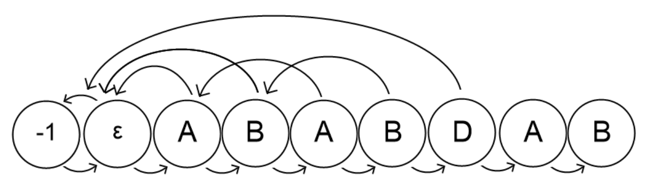

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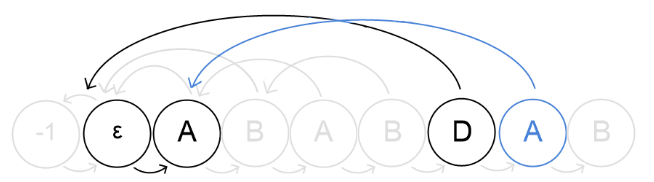

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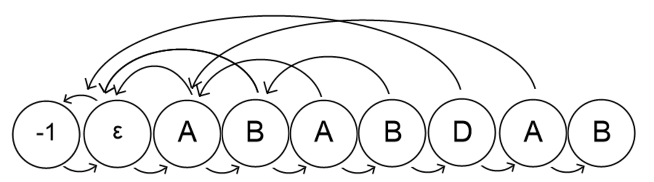

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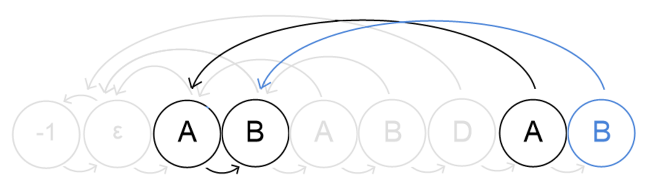

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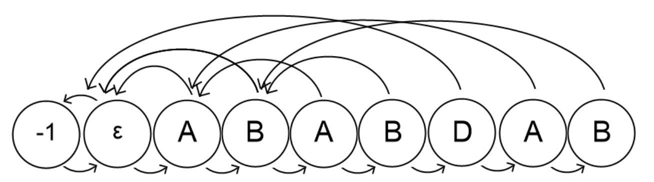

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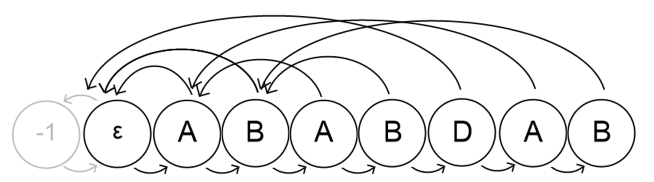

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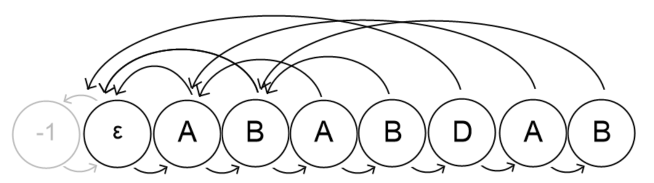

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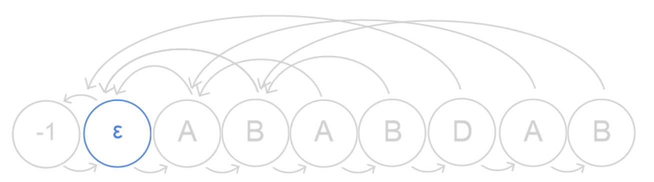
When we see char c, what's next state? Follow Fail arrow until next char matches or $k = -1$, then move right.

String matching with simplified DFA



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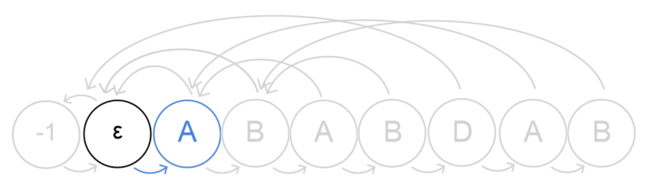

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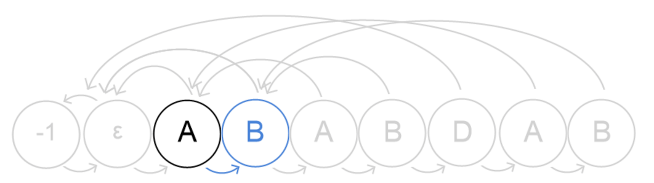

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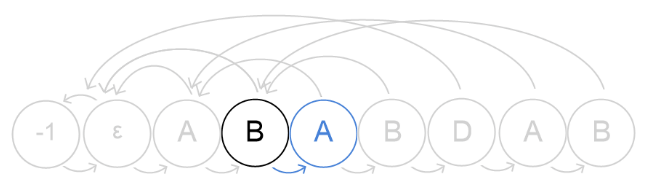

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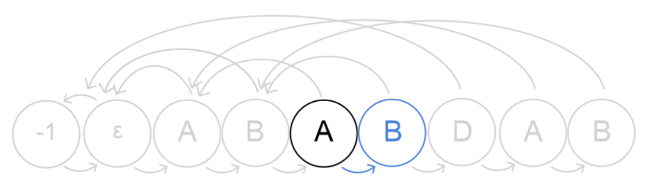

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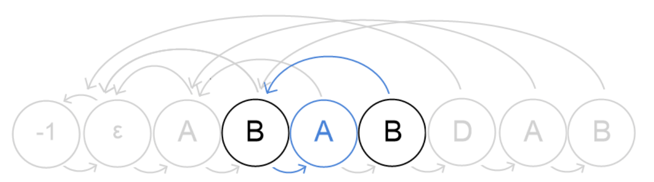

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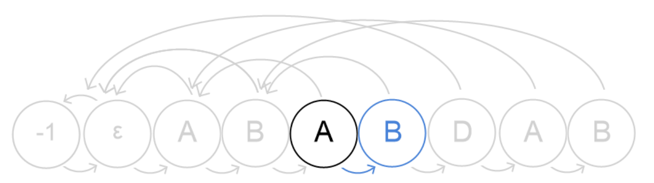

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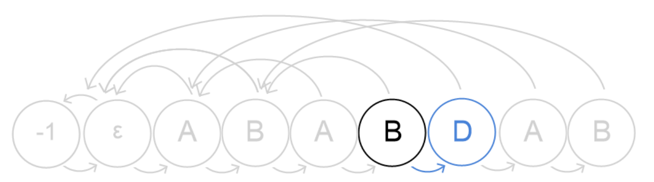

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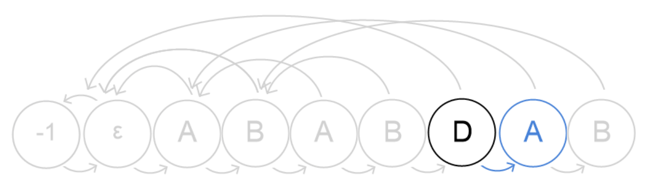

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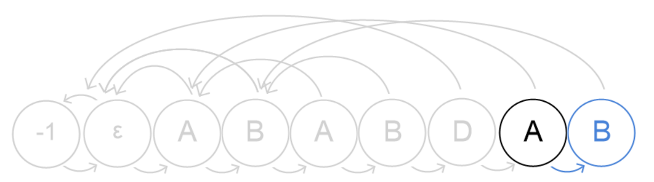

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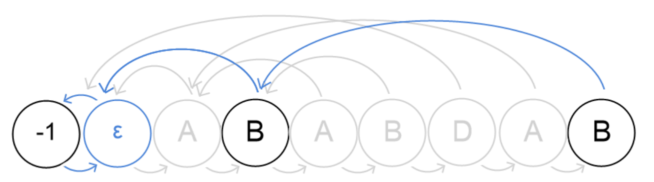

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The KMP Algorithm: Building the DFA

```
KMP_INIT(W):
  initialize array Fail of size |W|+1
  set Fail[0] = -1
  for i in 1 to |W|
    let nxt = Fail[i-1]
    while nxt >= 0 && W[nxt] != W[i-1]:
      nxt = Fail[nxt]
    set Fail[i] = nxt + 1
  return Fail
```

The KMP Algorithm: Matching

```
KMP_MATCH(Fail, W, S):
  initialize cur = 0, matches = empty list
  for i in 0 to |S| - 1
    while cur >= 0 && W[cur] != S[i]:
      cur = Fail[cur]
    cur = cur + 1
    if cur == |W|:
      add i - |W| + 1 to matches
      cur = Fail[cur]
  return matches
```

The KMP Algorithm: Time Complexity Analysis

Building the DFA for search string of size m

- To build next arrow, start at end point of previous arrow, take 0 or more back arrows, and 1 forward arrow.
- Every backward arrow move back at least 1 state
- No arrow moves past -1
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$\Rightarrow$ Time complexity is $O(m + n)$

Problem 1 – Wildcards

Find at least one occurrence of S_1 in string S_2.

Catch: a $*$ in string A matches any sequence of characters.

S_1	S_2	Match?
aa*b	aab	Yes
	aacdab	Yes
	caaccbd	Yes
	aacdaa	No
	accccb	No

Figure 1: Example of wildcard matching

How many $*$ can you handle?

Problem 1 – Solution

- Cut S_1 by $*$ into pieces $T_1, T_2, \ldots, T_k$
- Find first copy of T_1, then the first copy of T_2 after T_1, and so on
- Time complexity: still $O(m + n)$

Problem 2 – Wildcards Again

What if we only have one wild card, but it can appear anywhere?

S_1	S_2	Match?
computer	a computer	Yes
	cooooomputer	Yes
	compute0r	Yes
	cooompuuuter	No

Figure 2: Example of matching one wildcard anywhere

Problem 2 – Solution

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- Run KMP to find S_1 in S_2, and reverse(S_1) in reverse(S_2)
- Keep track of the KMP state at each character of S_2:
  - $A(k) =$ longest prefix of S_1 that matches suffix of $S_2[0 \ldots k]$
  - $B(k) =$ longest suffix of S_1 that matches prefix of $S_2[k \ldots n-1]$

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- Check if there are indices $i < j$ such that $A(i) + B(j) = m$
    - Can be done in linear time: iterate i from 0 to $n-1$, keep a boolean array of which values $A(i)$ we have seen so far, check if we have seen $m - B(i+1)$

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- Time complexity: $O(m+n)$

Trie

Problem: Implement `Map<String, Value>`

Suppose we store N strings of length $\leq M$ in a balanced BST.

Time complexity: $O(M \cdot \log N)$. Space complexity: $O(MN)$

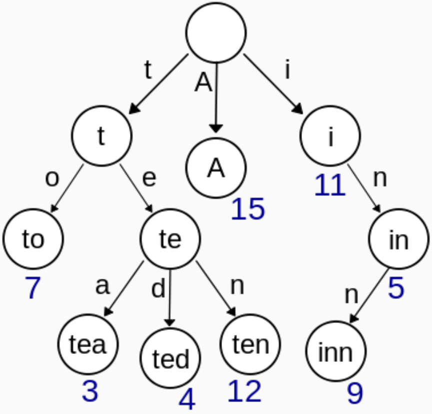
Can't do partial prefix match :(

Can we do better?

A Trie is a Tree!

Observation: there are only 26 letters in the alphabet, so we are storing lots of duplicates! Why not use the alphabet to form a tree?

Keys: to, tea, ted, ten, A, i, in, inn.



Source: *Wikipedia*

Trie Structure and Operations

```cpp
struct TrieNode {
  bool isWord;
  TrieNode *child[26];
  TrieNode() {
    isWord = false;
    memset(child, 0, sizeof child);
  }
};
```

Exercise: code up find(), insert(), delete(), isPrefixMatch()

Time complexity: $O(N)$ per operation where N = max string length
Space complexity: $O(NC)$ for C letters in alphabet

Problem 3 – Word Game

Two player game where you alternate turns adding a letter to a string.

At every turn, the string must be prefix of some word.

The person who adds the last letter of a word loses.

If you go first, can you win? Find the winning strategy!

Problem 3 – Solution

Perform tree dp on the trie of all words

- State: f(node) = can you win if you are here
- f(trie node that is a word) = false
- f(node) = true if f(child) = false for some child
- f(node) = false if f(child) = true for all child

Next Class

Aho Corasick
(i.e. KMP on a Trie)