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Notes

- Demetri Terzopoulos talk:
  Thursday, 4pm
  Dempster 310

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Back to Rigid Bodies

- Motivation - particle simulation doesn't cut it for large rigid objects
  - Especially useful for action in games and film (e.g. car dynamics, crashes, explosions)
- To recap:
  - Split our rigid body into chunks of matter, we look at each chunk as a simple particle
  - Rigid constraint: distances between particles have to stay constant
  - Thus position of a particle is a rotation + translation from "object space" into "world space"
  - We want to figure out what's happening with velocities, forces, …

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Rigid Motion

- Recall we map from object space position p_i of particle i to world space position x_i with
  $x_i=R(t)p_i+X(t)$
- Differentiate map w.r.t. time (using dot notation): $v_i = \dot{R}p_i + V$
- Invert map for p_i: $p_i = R^T(x_i - X)$
- Thus: $v_i = \dot{R}R^T(x_i - X) + V$
- 1st term: rotation, 2nd term: translation
  - Let's simplify the rotation

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Skew-Symmetry

- Differentiate $RR^T=\delta$ w.r.t. time:
  $$\dot{R}R^T + R\dot{R}^T = 0 \Rightarrow \dot{R}R^T = -\left(\dot{R}R^T\right)^T$$
- ♣ Skew-symmetric! Thus can write as:
  $$\dot{R}R^T = \begin{pmatrix} 0 & -\omega_2 & \omega_1 \\ \omega_2 & 0 & -\omega_0 \\ -\omega_1 & \omega_0 & 0 \end{pmatrix}$$
- ♣ Call this matrix ω^* (built from a vector ω)
  $$\dot{R}R^T = \omega^* \quad \Rightarrow \quad \dot{R} = \omega^* R$$

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The cross-product matrix

- Note that:

$$\omega^* x = \begin{pmatrix} 0 & -\omega_2 & \omega_1 \\ \omega_2 & 0 & -\omega_0 \\ -\omega_1 & \omega_0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \omega_1 x_2 - \omega_2 x_1 \\ \omega_2 x_0 - \omega_0 x_2 \\ \omega_0 x_1 - \omega_1 x_0 \end{pmatrix} = \omega \times x$$

- So we have:

$$v_i = \omega \times (x_i - X) + V$$

♣ ω is the angular velocity of the object

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Angular velocity

- Recall:
  - |ω| is the speed of rotation (radians per second)
  - ♣ω points along the axis of rotation (which in this case passes through the point X)
  - ♣Convince yourself this makes sense with the properties of the cross-product

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Force

- Take another time derivative to get acceleration:

$$a_i = \dot{v}_i = \ddot{R} p_i + A$$

- Use F=ma, sum up net force on system:

$$\sum_i F_i = \sum_i m_i a_i = \sum_i m_i (\ddot{R} p_i + A)$$
$$= \ddot{R} \sum_i m_i p_i + A \sum_i m_i$$

- Let the total mass be $M = \sum_i m_i$
- How to simplify the other term?

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Centre of Mass

- Let's pick a new object space position:

$$p_i^{new} = p_i - \frac{\sum_j m_j p_j}{M}$$

  - The mass-weighted average of the positions is the centre of mass
  - We translated the centre of mass (in object space) to the point 0
- Now:

$$\sum_i m_i p_i = 0$$

Force equation

- So now, assuming we've set up object space right (centre of mass at 0), F=MA
- If there are no external forces, have F=0
  - Internal forces must balance out, opposite and equal
  - Thus A=0, thus V=constant
- If there are external forces, can integrate position of object just like a regular particle!

What about R?

- How does orientation change?
- Think about internal forces keeping the particles in the rigid configuration
  - Conceptual model: very stiff spring between every pair of particles, maintaining the rest length
- So $F_i = \sum_j f_{ij}$ where f_{ij} is force on i due to j
- Of course $f_{ij}+f_{ji}=0$
- Also: f_{ij} is in the direction of x_i-x_j
  - Thus $\left(x_i - x_j\right)\times f_{ij} = 0$

Net Torque

- Play around: $\left((x_i - X)-(x_j - X)\right)\times f_{ij} = 0$

$$\left(x_i - X\right)\times f_{ij} = \left(x_j - X\right)\times f_{ij}$$
$$= -\left(x_j - X\right)\times f_{ji}$$

- Sum both sides (look for net force)

$$\sum_{i,j}\left(x_i - X\right)\times f_{ij} = -\sum_{i,j}\left(x_j - X\right)\times f_{ji}$$
$$\sum_{i}\left(x_i - X\right)\times F_i = -\sum_{j}\left(x_j - X\right)\times F_j$$
$$= 0$$

- The expression we just computed=0 is the net torque on the object

Torque

- The torque of a force applied to a point is

$$\tau_i = \left(x_i - X\right)\times F_i$$

- The net torque due to internal forces is 0
- [geometry of torque: at CM, with opposite equal force elsewhere]
- Torque obviously has something to do with rotation
- How do we get formula for change in angular velocity?

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Angular Momentum

- Use F=ma in definition of torque:

$$\tau_i = (x_i - X) \times m_i a_i$$
$$= \tfrac{d}{dt}\left[m_i (x_i - X) \times v_i\right]$$

- force=rate of change of linear momentum, torque=rate of change of angular momentum
- The total angular momentum of the object is

$$L = \sum_i m_i (x_i - X) \times v_i$$
$$= \sum_i m_i (x_i - X) \times (v_i - V)$$

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Getting to ω

- Recall $v_i - V = \omega \times (x_i - X)$
- Plug this into angular momentum:

$$L = \sum_i m_i (x_i - X) \times (\omega \times (x_i - X))$$
$$= -\sum_i m_i (x_i - X) \times ((x_i - X) \times \omega)$$
$$= -\sum_i m_i (x_i - X)^* (x_i - X)^* \omega$$
$$= \underbrace{\left(\sum_i m_i (x_i - X)^{*T} (x_i - X)^*\right)}_{I(t)} \omega$$

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Inertia Tensor

- I(t) is the inertia tensor
- Kind of like "angular mass"
- Linear momentum is mv
- Angular momentum is L=I(t)ω

♣ Or we can go the other way: $\omega = I(t)^{-1} L$

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Equations of Motion

$$\tfrac{d}{dt} V = F/M \qquad \tfrac{d}{dt} L = \mathrm{T}$$
$$\tfrac{d}{dt} X = V \qquad \omega = I(t)^{-1} L$$
$$\tfrac{d}{dt} R = \omega^* R$$

In the absence of external forces F=0, T=0

Reminder

- Before going on:
- Remember that this all boils down to particles
  - Mass, position, velocity, (linear) momentum, force are fundamental
  - Inertia tensor, orientation, angular velocity, angular momentum, torque are just abstractions
  - Don't get too puzzled about interpretation of torque for example: it's just a mathematical convenience