CPSC 424
Review 3 (meshes ++)

Tensor Product Surfaces

More General Parametric Surfaces

- Use basis functions like for curves
- Apply independently to parametric directions s and t
- Works for arbitrary basis

Example:

- Bézier curve:

$$F(t) = \sum_{i=0}^{m} B_i^m(t) \cdot \mathbf{b}_i$$

- Tensor product Bézier patch:

$$F(s,t) = \sum_{i=0}^{m_s} \sum_{j=0}^{m_t} B_i^{m_s}(s) \cdot B_j^{m_t}(t) \cdot \mathbf{b}_{i,j}$$

Tensor Product Surfaces

Continuity

- Two patches

$$F(s,t):[s_0,s_1]\times[t_0,t_1],$$
$$G(s,t):[s_1,s_2]\times[t_0,t_1]$$

- are C^k continuous if <u>for all t</u>

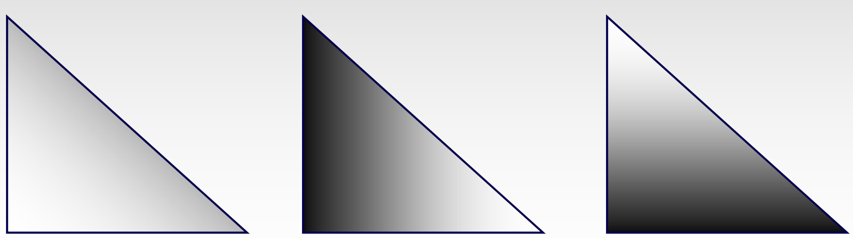
$$F^{(l)}(s,t)=G^{(l)}(s,t); l\leq k$$

- Same for s
- Special case – two patches sharing one corner

Triangles

Barycentric Coordinates:

$$\mathbf{p}=\alpha\mathbf{v}_0+\beta\mathbf{v}_1+\gamma\mathbf{v}_2;\alpha+\beta+\gamma=1$$



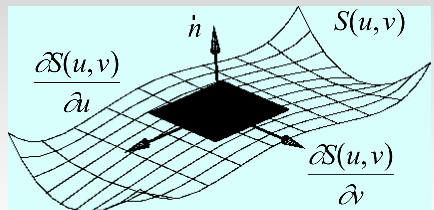
Surfaces – differential geometry

Tangent plane to surface S(u,v) is spanned by two partials of S:

$$\frac{\partial S(u,v)}{\partial u} \quad \frac{\partial S(u,v)}{\partial v}$$

Normal ***to surface***

$$\vec{n} = \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}$$



- perpendicular to tangent plane

Any vector in tangent plane is tangential to S(u,v)

Curvature

Normal curvature ***of surface is defined for each tangential direction***

Principal curvatures **Kmin & Kmax*****: maximum and minimum of normal curvature***

- Correspond to two **orthogonal** tangent directions
  - *Principal directions*
- Not necessarily partial derivative directions
- Independent of parameterization

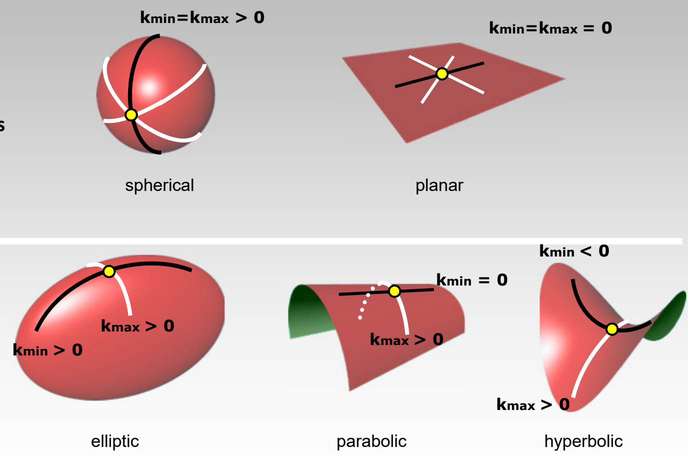
3D Curvature

Isotropic

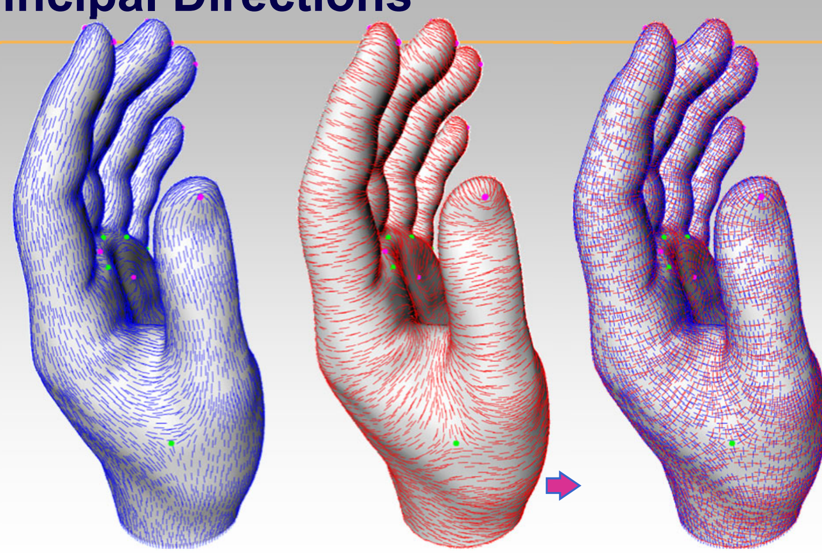
Equal in all directions

Anisotropic

2 distinct principal directions



Principal Directions



min curvature max curvature

Curvature

Typical measures:

- ***Gaussian*** curvature

$$K = k_{\min} k_{\max}$$

- ***Mean*** curvature

$$H = \frac{k_{\min} + k_{\max}}{2}$$

Clicker questions:

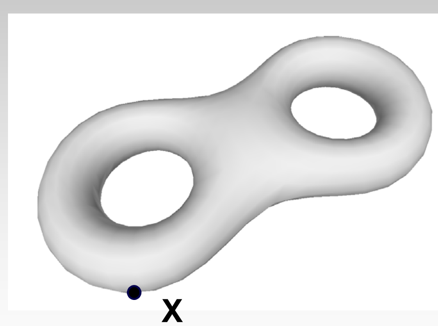
Which type of surface locally is point X?

A. Parabolic

B. Hyperbolic

C. Elliptic (non-isotropic)

D. Isotropic



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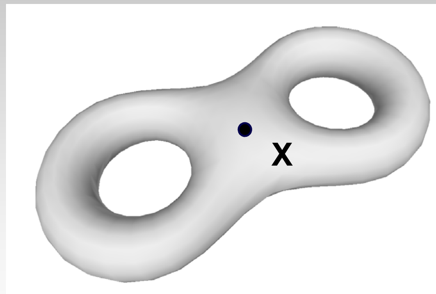

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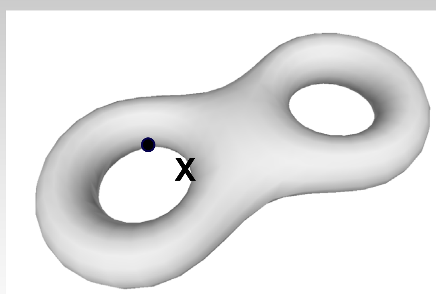
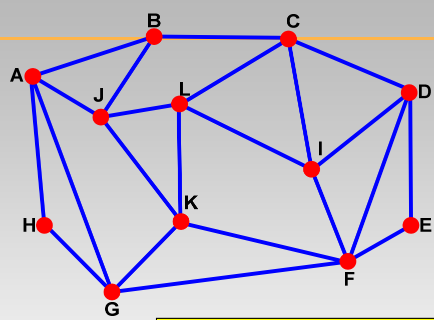


Standard Graph Definitions



G = <**V**,**E**>
V = vertices =
{A,B,C,D,E,F,G,H,I,J,K,L}
E = edges =
{(A,B),(B,C),(C,D),(D,E),(E,F),(F,G),
(G,H),(H,A),(A,J),(A,G),(B,J),(K,F),
(C,L),(C,I),(D,I),(D,F),(F,I),(G,K),
(J,L),(J,K),(K,L),(L,I)}

Vertex degree (valence) = number of edges incident on vertex
deg(J) = 4, deg(H) = 2

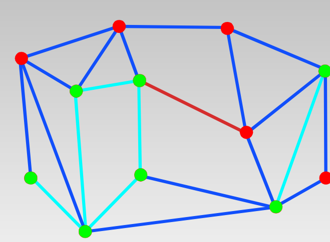
Face: cycle of vertices/edges which cannot be shortened
F = faces =
{(A,H,G),(A,J,K,G),(B,A,J),(B,C,L,J),(C,I,L),(C,D,I),
(D,E,F),(D,I,F),(L,I,F,K),(L,J,K),(K,F,G)}

Connectivity

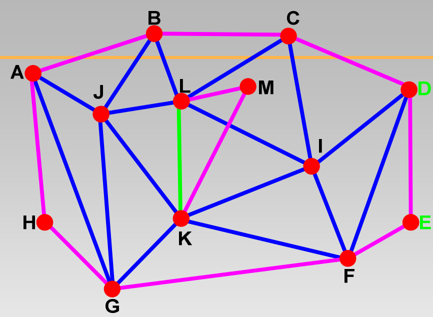
Graph is ***connected*** if there is a path of edges connecting every two vertices

Graph **G'**=<**V'**,**E'**> is a ***subgraph*** of graph **G**=<**V**,**E**> if **V'** is a subset of **V** and **E'** is the subset of **E** incident on **V'**

Connected component of a graph: maximal connected subgraph



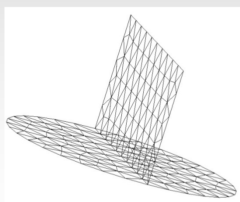
Meshes



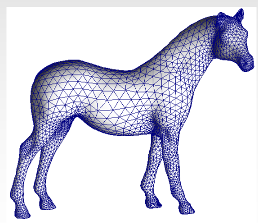
Mesh: graph embedded in R^3

Boundary edge: adjacent to exactly *one* face
Regular edge: adjacent to exactly *two* faces
Singular edge: adjacent to more than two faces

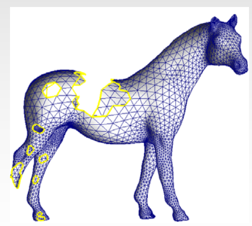
Closed mesh: mesh with no boundary edges
Manifold mesh: mesh with no singular edges



Non-Manifold

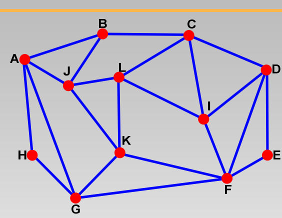


Closed Manifold



Open Manifold

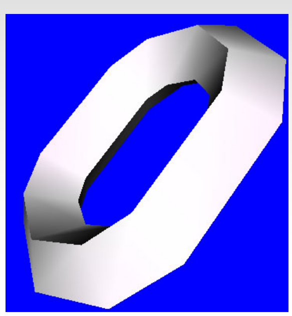
Topology

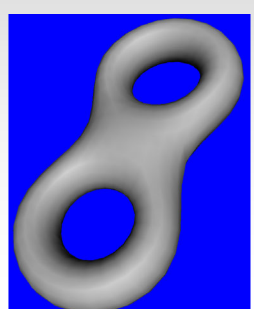


v =12
f = 14
e = 25
c = 1
g = 0
b = 1

Genus of graph: *half* of *maximal* number of closed paths that do *not* disconnect the graph (number of "holes")

Genus(sphere)= 0
Genus(torus) = 1





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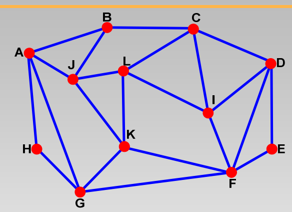


v =12
f = 11
e = 22
c = 1
g = 0
b = 1

Euler-Poincare Formula

v+f-e = 2(c-g)-b

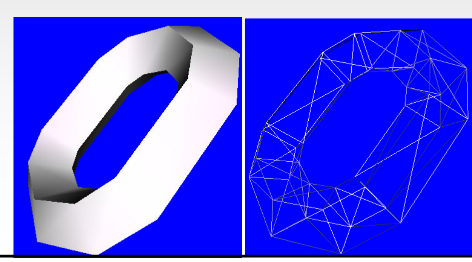
v = # vertices c = # conn. comp
f = # faces g = genus
e = # edges b = # boundaries

Exercises

Theorem: Average vertex degree in closed manifold triangle mesh is ~6

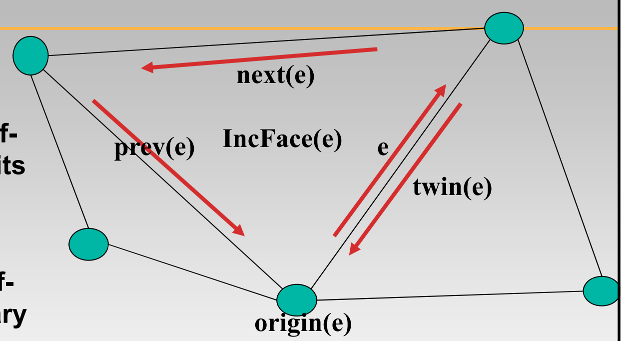
Proof: In such a mesh, f = 2e/3
By Euler's formula: v+2e/3-e = 2-2g
hence e = 3(v-2+2g) and f = 2(v-2+2g)

So Average(deg) = 2e/v = 6(v-2+2g)/v
~ 6 for large v



Half-Edge Data Structure

- **Vertex record:**
  - **Coordinates**
  - **Pointer to one half-edge that has v as its origin**
- **Face record:**
  - **Pointer to one half-edge on its boundary**



Half-edge record:

- Pointer to its origin, origin(e)
- Pointer to its twin half-edge, twin(e)
- Pointer to the face it bounds, IncidentFace(e) (face lies to left of e when traversed from origin to destination)
- Next and previous edge on boundary of IncidentFace(e)

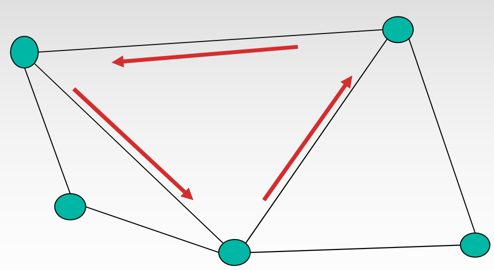
Half-Edge Data Structure (cont.)

Operations supported:

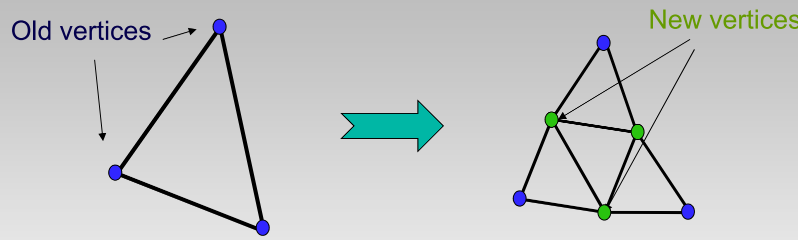
- Walk around boundary of given face
- Visit all edges incident to vertex v

Queries:

- Most queries are O(1)



Triangular subdivision



Each face replaced by 4 new faces

Two kinds of new vertices:

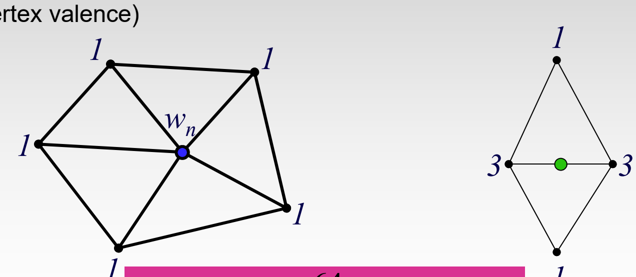
- Green vertices are associated with old edges
- Blue vertices are associated with old vertices

Loop's scheme

New vertex is weighted average of old vertices

List of weights called subdivision mask or stencil

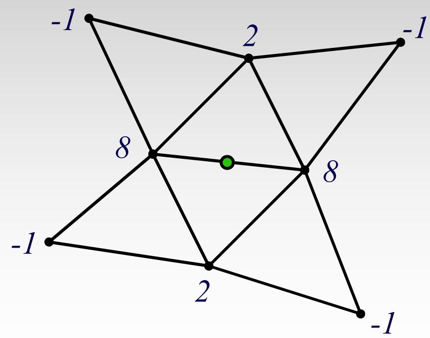
- Rule for new blue vertices (n – vertex valence)
- Rule for new green vertices



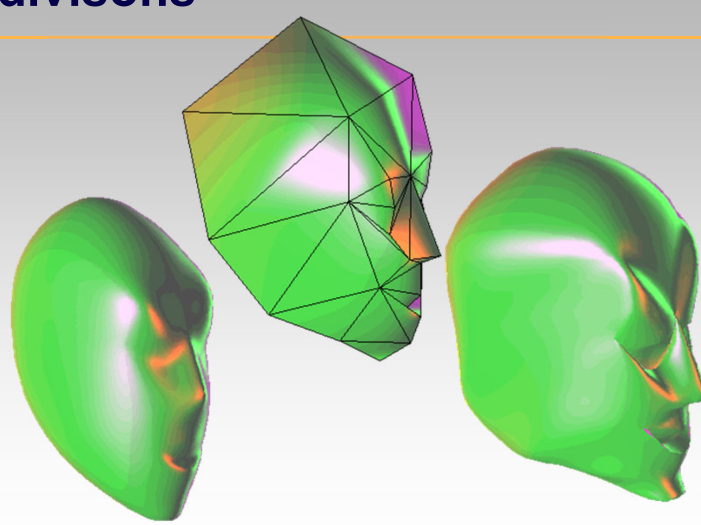
$$w_n = \frac{64n}{40 - (3 + 2\cos(2\pi/n))^2} - n$$

Butterfly Scheme

- Interpolatory scheme
- New blue vertices inherit location of old vertices
- New green vertices calculated by following stencil:



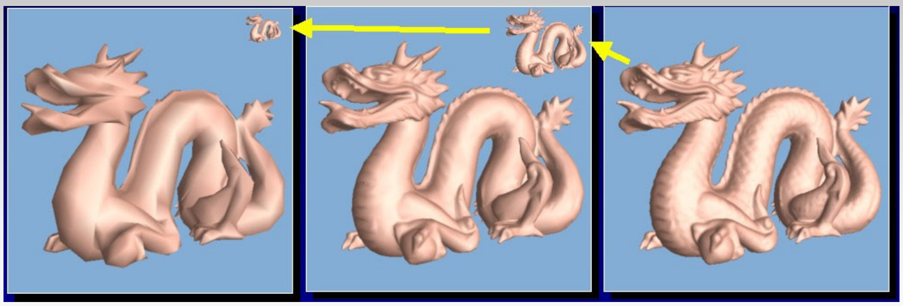
Subdivisons



Level of Detail (LOD)

Refined mesh for close objects

Simplified mesh for far



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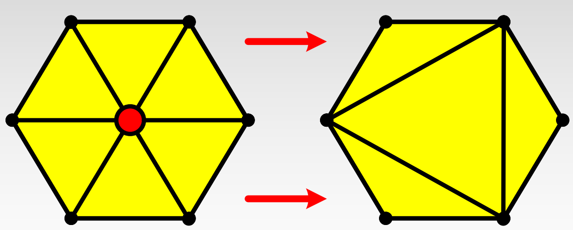
Simplification Operations (1)

Decimation

- Vertex removal:
  - $v \leftarrow v\text{-}1$
  - $f \leftarrow f\text{-}2$



Remaining vertices - subset of original vertex set

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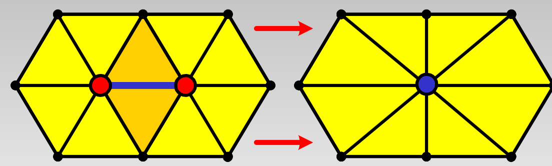
Simplification Operations (2)

Decimation

- Edge collapse
  - $v \leftarrow v\text{-}1$
  - $f \leftarrow f\text{-}2$



Vertices may move

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The Basic Algorithm

Repeat

- Select the element with minimal error
- Perform simplification operation (remove/contract)
- Update error (local/global)

Until mesh size / quality is achieved

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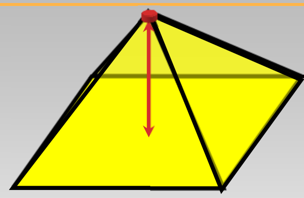
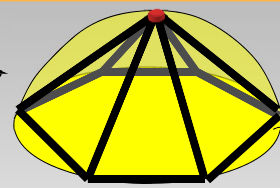
Simplification Error Metrics



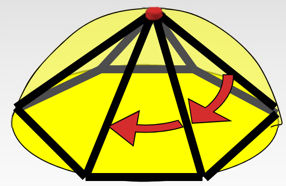


Measures

- Distance to plane
- Curvature

Usually approximated

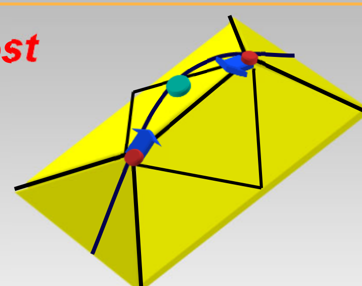
- Average plane
- Discrete curvature



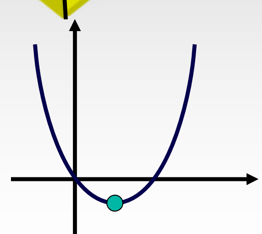
$\Sigma\alpha / 2\pi$

Distance Metric: Quadrics

Choose point closest to set of planes (triangles)



Sum of squared distances to set of planes is quadratic $\Rightarrow$ has a minimum



Quadrics

Plane

- $Ax + By + Cz + D = 0$, where $A^2 + B^2 + C^2 = 1$
- $p = [A, B, C, D]$, $v = [x, y, z, 1]$, $v\,p^T = 0$

Quadratic distance between v and p:

$$\Delta_p(v) = (v\,p^T)^2 = (v\,p^T)(p\,v^T) = v\,(p^T p)\,v^T = v\,K_P\,v^T$$

$$K_P = \begin{bmatrix} A^2 & AB & AC & AD \\ AB & B^2 & BC & BD \\ AC & BC & C^2 & CD \\ AD & BD & CD & D^2 \end{bmatrix}$$

Distance to Set of Planes

$$\Delta(v) = \sum_{p \in planes(v)} \Delta_p(v) = \sum_{p \in planes(v)} (v\,K_p\,v^T) = v\,\Big(\sum_{p \in planes(v)} K_p\Big)\,v^T = v\,Q_v\,v^T$$

After v_1, v_2 are contracted to v,
$Q_v \leftarrow Q_{v1}+Q_{v2}$

Pseudo-global

All original planes persist during the entire simplification process