CPSC 424
Review 2 (curves & surfaces)

Curves: Review

Curves in 2D and 3D

- Implicit vs. Explicit vs. Parametric curves
- Bézier curves, de Casteljau algorithm
- Continuity
- B-Splines
- Subdivision Curves

Bézier Curves

Definition:

- Bézier curve is a polynomial curve that uses **Bernstein polynomials** as basis

$$F(t) = \sum_{i=0}^{m} \mathbf{b}_i B_i^m(t)$$

- b_i are called control points of Bézier curve
- Control polygon obtained by connecting control points with line segments

Bernstein Polynomials

$$B_i^m(t) := \binom{m}{i} t^i (1-t)^{m-i}; i = 0..m; t \in [0,1],$$

$$\binom{m}{i} = \frac{m!}{(m-i)!i!}$$

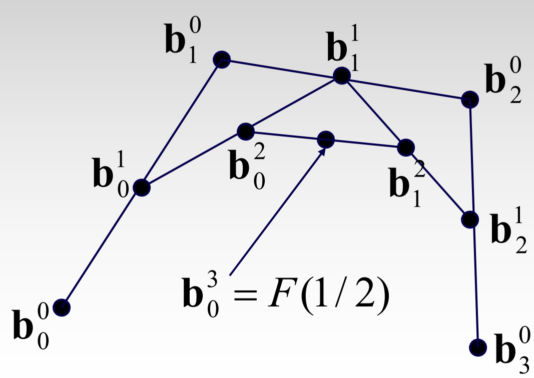
Properties of Bézier Curves

- Endpoints b_0 and b_m of control polygon interpolated & corresponding parameter values are t=0 and t=1
- Bézier curve is tangential to control polygon at endpoints
- Curve lies within convex hull of control points
- Curve is *affine invariant*
- There is a fast, recursive evaluation algorithm – de Casteljau algorithm

De Casteljau Algorithm

Graphical Interpretation:

- Determine point $F(1/2)$ for the cubic Bézier curve given by the following four points:



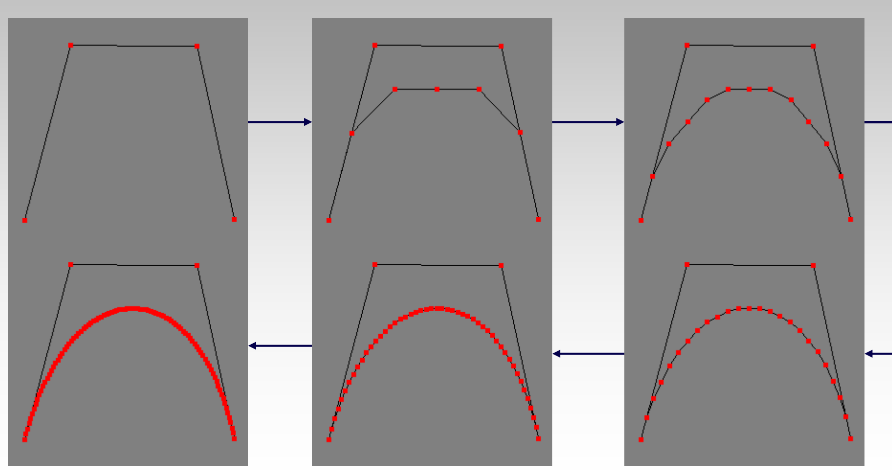
Observation

De Casteljau generates 2 new control polygons!

- For parameter interval [0,1/2], and [1/2,1]
- Can be used to recursively subdivide control polygon

Bezier Subdivision

Cubic case:



Derivatives of Bézier Curves

- The derivative of a Bézier curve

is given as

$$F(t) := \sum_{i=0}^{m} B_i^m(t) \cdot \mathbf{b}_i$$

$$F'(t) := m \cdot \sum_{i=0}^{m-1} B_i^{m-1}(t) \cdot (\mathbf{b}_{i+1} - \mathbf{b}_i)$$

Continuity

Def:

- A curve $F(t)$ is called C^k-continuous if its k^{th} derivative $F^{(k)}(t)$ exists (i.e. is continuous) everywhere

Note:

- Polynomial curves are infinitely continuous

Def:

- Two curve segments $F(t)$ and $G(t)$ are called C^k-continuous at t_0 if their first k derivatives match at t_0

Splines

Concept:

- Provide local control by piecing together multiple (polynomial) curves in smooth fashion
- This is called Spline

Bezier spline:

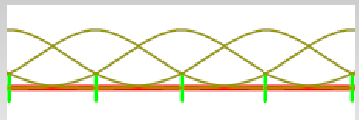
- sequence of Bezier curves (joined at different levels of continuity)

Bezier Spline Continuity

- ***C^0 :*** share end control points $b_m = b'_0$
- ***C^1 :*** $b_m - b_{m-1} = b'_1 - b'_0$
- ***G^1 :*** $b_m - b_{m-1}$ collinear to $b'_1 - b'_0$

B-Splines

Idea: Generate basis where functions are continuous cross domains



Control point controls set of basis functions (to preserve continuity)

Alternative view: continuous basis functions defined on several domains

B-Splines

Direct recursion formula:

$$N_i^0(t) = \begin{cases} 1 & ; u_i \le t < u_{i+1} \\ 0 & ; \text{else} \end{cases}$$

$$N_i^l(t) = \frac{t - u_i}{u_{i+l} - u_i} \cdot N_i^{l-1}(t) + \frac{u_{i+l+1} - t}{u_{i+l+1} - u_{i+1}} \cdot N_{i+1}^{l-1}(t)$$

Note:

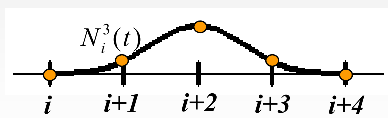
- Not an affine combination

Uniform Cubic B-Spline Curves

Definition

$$C(t) = \sum_{i=0}^{n-1} P_i N_i^3(t) \quad t \in [3,n]$$

$$N_i^3(t) = \begin{cases} r^3/6 & r = t-i & t \in [i, i+1] \\ (-3r^3 + 3r^2 + 3r + 1)/6 & r = t-i-1 & t \in [i+1, i+2] \\ (3r^3 - 6r^2 + 4)/6 & r = t-i-2 & t \in [i+2, i+3] \\ (1-r)^3/6 & r = t-i-3 & t \in [i+3, i+4] \end{cases}$$



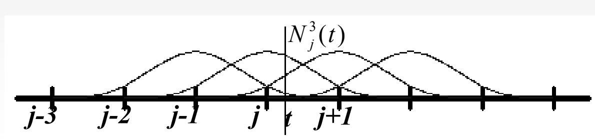
Uniform Cubic B-Spline Curves

For any t ∈ [3,n] $\sum_{i=j-3}^{j} N_i^3(t) = 1$

For any t ∈ [j,j+1] only 4 basis functions are non zero

$$\sum_{i=0}^{n-1} N_i^3(t) = 1$$

Any point on cubic B-Spline is affine combination of at most 4 control points



Boundary Conditions for B-Splines

B-Splines do not interpolate any control points

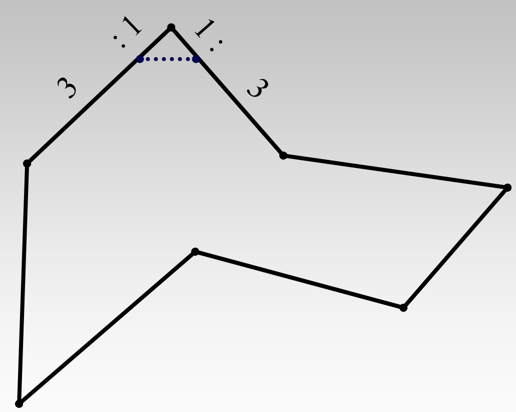
- in particular end points
- Can achieve interpolation by replicating control points (or knots)

NURBs

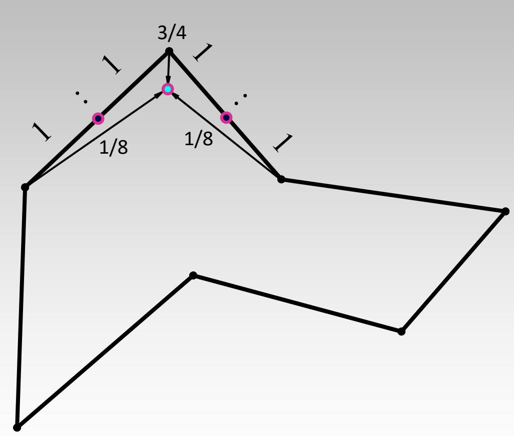
- B-Spline (B-spline basis)
- Non-Uniform – different interval lengths (knots)
- Rational – rational basis functions

$$C(t) = \frac{\sum_{i=0}^{n-1} w_i P_i N_i^3(t)}{\sum_{i=0}^{n-1} w_i N_i^3(t)} \quad t \in [3, n]$$

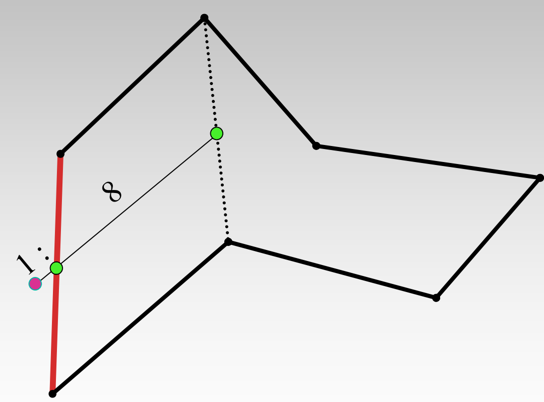
Subdivision: Corner Cutting – Chaikin Algorithm



Cubic B-Spline (corner cutting)



The 4-point scheme



Proving scheme works

Proving scheme works:

- **Convergence**
- Degree of continuity
- Affine invariance
  - *As long as weights sum to 1*

Subdivision Matrix

Example: Chaikin subdivision

$$\begin{pmatrix} P_0^i \\ P_1^i \\ P_2^i \\ P_3^i \end{pmatrix} = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 \\ 0 & 3/4 & 1/4 & 0 \\ 0 & 1/4 & 3/4 & 0 \\ 0 & 0 & 3/4 & 1/4 \end{pmatrix}^i \begin{pmatrix} P_0^0 \\ P_1^0 \\ P_2^0 \\ P_3^0 \end{pmatrix}$$

Syllabus

Curves in 2D and 3D

- …
- Subdivision Curves

Properties of Curves and Surfaces

- **Differential Geometry**:
  - *arc length*
  - *curvature*
  - *Fresnet frame*

Surfaces

Regularity

Definition:

- Differentiable parametric curve $F(t):[a,b] \mapsto \mathcal{R}^3$ is called regular if

$$F'(t) \neq 0, \forall t \in [a,b]$$

- (I.e. if the tangent vector is not 0 anywhere)

Note:

- Bézier curves not necessarily regular…

Equivalence/Reparameterization

Definition

- Two regular curves

$$F(t):[a,b] \mapsto \mathcal{R}^3 \qquad G(t):[c,d] \mapsto \mathcal{R}^3$$

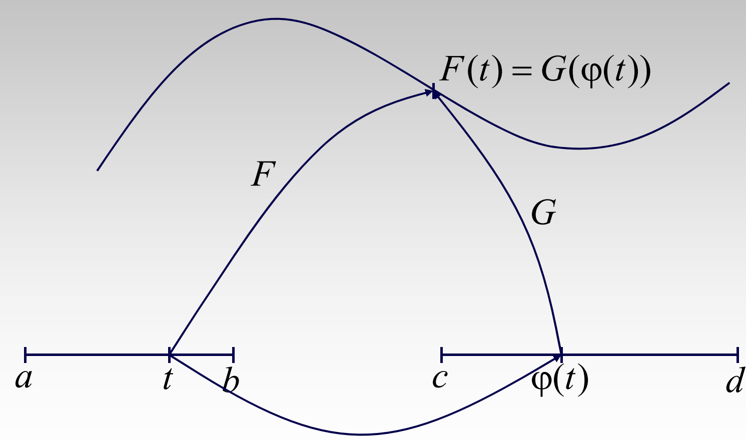
are geometrically equivalent $F \simeq G$ if there is a strictly monotonic, differentiable *reparameterization* function

$$\varphi(t):[a,b] \mapsto [c,d]$$

with

$$F(t) = G(\varphi(t))$$

Equivalence/Reparameterization



Clicker Question

Are the curves

F(t) = (t,t) t in [0,1] and G(t) = (t/3,t/3) t in [0,3]

geometrically equivalent?

- A. Yes
- B. No
- Not enough information

Arc Length

Definition

- Arc length of regular curve $F(t):[a,b] \mapsto \mathcal{R}^3$ given as

$$s(t) := \int_a^t \| F'(\tilde{t}) \| \, d\tilde{t}$$

Parameterization by arc length

$$G(s) \text{ with } G(s(t)) = F(t)$$

- Note: this is a **canonical** representation for any curve
- Point is traveling along G with constant speed 1

Curvature

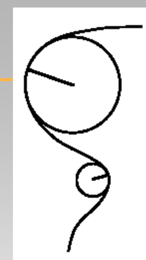
Definition

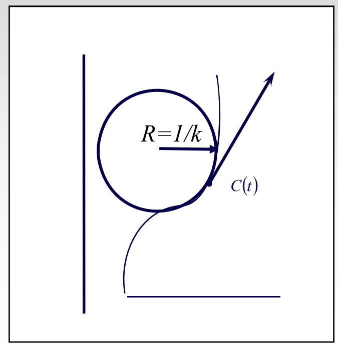
- Let G be a curve parameterized by arc length
- We introduce the following terms:
  - *Unit tangent* $T(s) := G'(s)$
  - *Curvature vector* $K(s) := G''(s)$
  - *Curvature* $\kappa(s) := \| K(s) \|$
  - *Principal normal* $N(s) := K(s) / \kappa(s)$
  - *Binormal* $B(s) := T(s) \times N(s)$

Curvature

Corresponds to radius of osculating circle **R=1/k**

Measure curve bending





Frenet Frame

Theorem:

- Curvature vector and tangent vector are perpendicular:

$$K(s) \perp T(s)$$

Note:

- Therefore, T, N, and B form an orthonormal coordinate frame
- This is called the Frenet Frame

Torsion

With the same argument we get

$$B'(s) = \tau(s) \cdot N(s)$$

Note:

- B' is the torsion vector
- τ is the torsion, and indicates how much the curve twists out of the plane (τ=0 means perfectly planar)

Fundamental Theorem of Curves

Theorem:

- For given functions $\kappa(s)$, $\tau(s)$ there exists exactly one (except for rotations and translations) unique curve that is parameterized by arc length and has curvature $\kappa(s)$, and torsion $\tau(s)$

Proof:

- Quite complex, see for example
  - *Da Carmo*
    Differential Geometry of Curves and Surfaces

Geometric Continuity

Definition:

- Two curves

$$F_1(t):[a,b] \mapsto \mathcal{R}^3, F_2(t):[b,c] \mapsto \mathcal{R}^3$$

are G^k-continuous (geometrically continuous of degree k), if there are reparameterizations

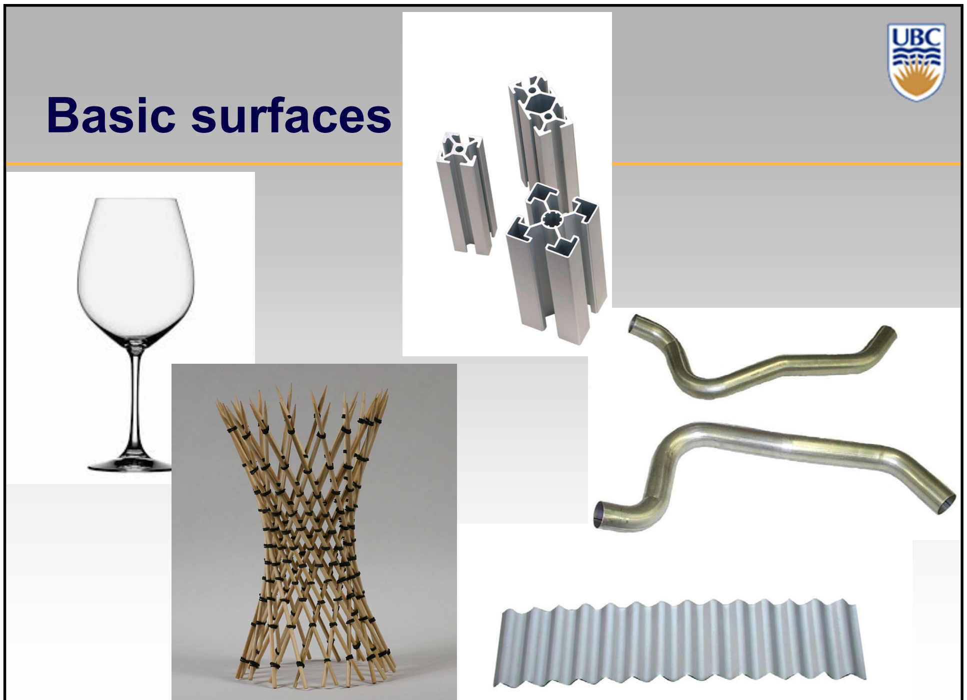
$$G_1(t) \cong F_1(t) \text{ and } G_2(t) \cong F_2(t)$$

that are C^k continuous, i.e.:

$$G_1^l(t) = G_2^l(t), l = 0 \ldots k$$

at shared parameter interval endpoint

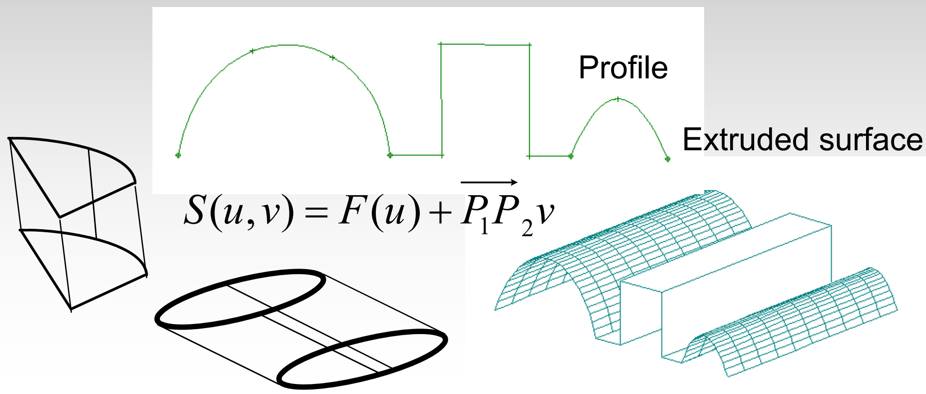
Basic surfaces



Extrusion

Concept:

- Move a curve ("profile") along a line segment
- The union of all points visited defines the surface



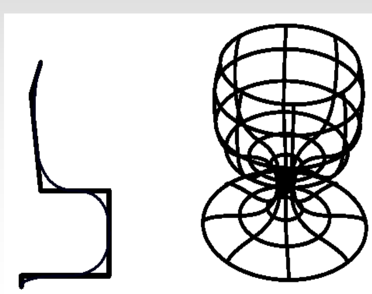
$$S(u,v) = F(u) + \overrightarrow{P_1 P_2}\, v$$

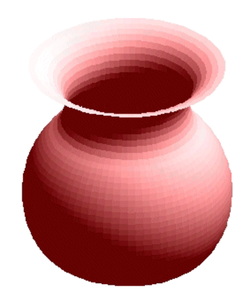
Surfaces of Revolution

Concept:

- Rotate profile curve around an axis
- $R(v)$ rotation matrix (v in $[0,2\pi]$)

$$S(u,v) = R(v)F(u)$$

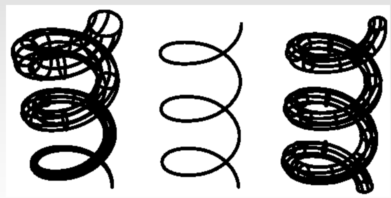




Sweeping

Concept:

- Generalize extrusion & revolution - sweep along arbitrary curve
- To orient profile at any point
  - *user specified*
  - *use Fresnet frame*

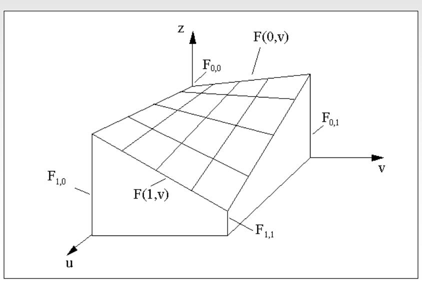


Bilinear Patches

Bilinear interpolation of 4 3D points - 2D analog of 1D linear interpolation between 2 points in the plane

Given $P_{00}, P_{01}, P_{10}, P_{11}$ - associated parametric bilinear surface for $u, v \in [0,1]$ is:

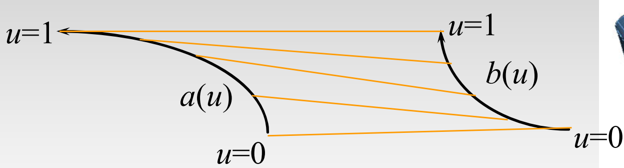
$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$

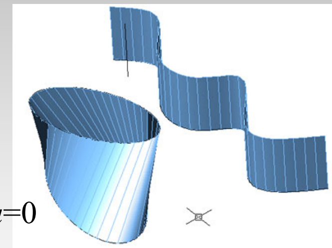


Ruled Surfaces

- Given two curves a(t) and b(t) corresponding ruled surface is constructed by connecting curves with straight lines

$$S(u,v)=va(u)+(1-v)b(u)$$



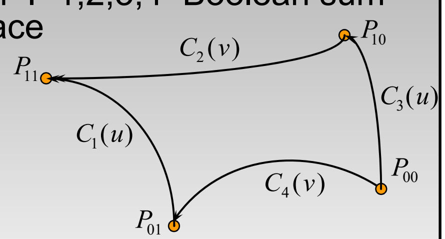


Questions:

- When is a ruled surface a bilinear patch ?
- When is a bilinear patch a ruled surface ?

Boolean Sum/Coons Patch (1967)

Given four connected curves Ci I=1,2,3,4 Boolean sum S(u,v) fills the interior with surface



$$S_1(u,v) = vC_1(u) + (1-v)C_3(u)$$

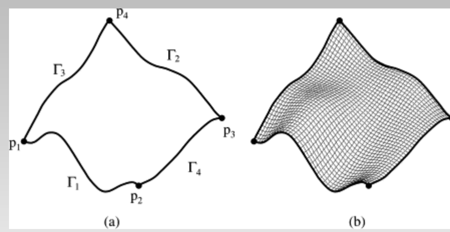
$$S_2(u,v) = uC_2(v) + (1-u)C_4(v)$$

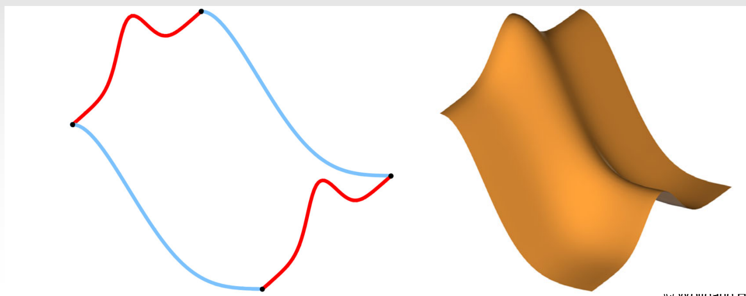
$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$

$$S(u,v) = S_1(u,v) + S_2(u,v) - P(u,v)$$

S(u,v) coincides with Ci along its boundaries

Examples





Tensor Product Surfaces

More General Parametric Surfaces

- Use basis functions like for curves
- Apply independently to parametric directions s and t
- Works for arbitrary basis

Example:

- Bézier curve:

$$F(t) = \sum_{i=0}^{m} B_i^m(t) \cdot \mathbf{b}_i$$

- Tensor product Bézier patch:

$$F(s,t) = \sum_{i=0}^{m_s} \sum_{j=0}^{m_t} B_i^{m_s}(s) \cdot B_j^{m_t}(t) \cdot \mathbf{b}_{i,j}$$

Clicker question

What kind of surface best describes the shape on the right?

A. Extrusion

B. Revolution

C. Sweep

D. Coons Patch

E. Ruled Surface

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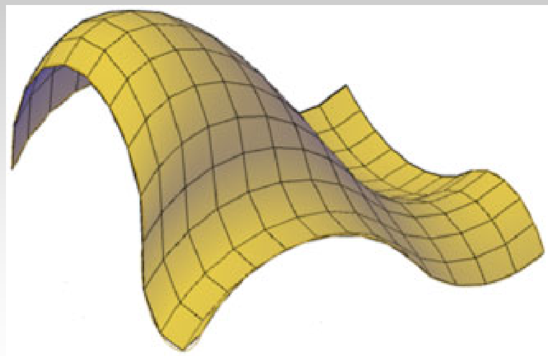

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Example:

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- Tensor product Bézier patch:

$$F(s,t) = \sum_{i=0}^{m_s} \sum_{j=0}^{m_t} B_i^{m_s}(s) \cdot B_j^{m_t}(t) \cdot \mathbf{b}_{i,j}$$

Tensor Product Surfaces

Continuity

- Two patches

$$F(s,t):[s_0,s_1]\times[t_0,t_1],$$
$$G(s,t):[s_1,s_2]\times[t_0,t_1]$$

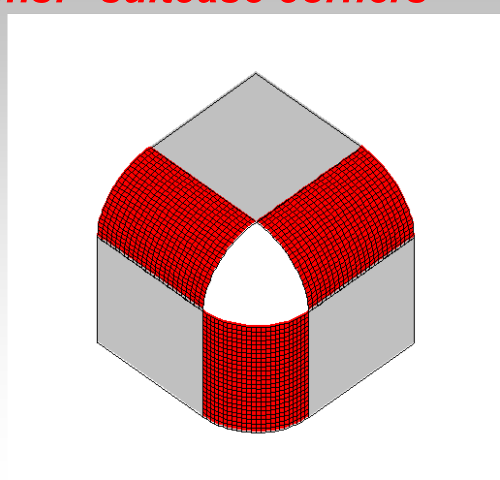
- are C^k continuous if for all t

$$F^{(l)}(s,t)=G^{(l)}(s,t);l\le k$$

- Same for s
- Special case – two patches sharing one corner

Tensor Product Surfaces

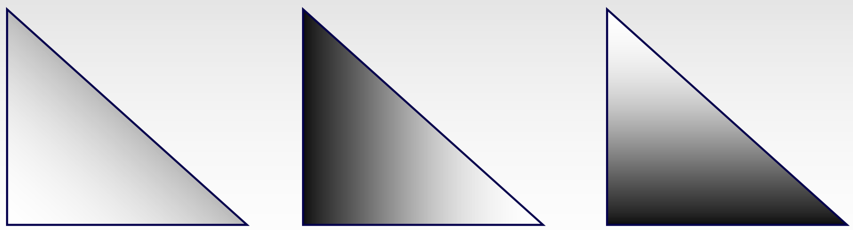
Limitations: "suitcase corners"



Bézier Triangles

Barycentric Coordinates:

$$\mathbf{p} = \alpha\mathbf{v}_0 + \beta\mathbf{v}_1 + \gamma\mathbf{v}_2 ; \alpha + \beta + \gamma = 1$$



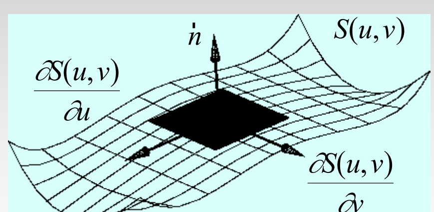
Surfaces – differential geometry

Tangent plane to surface S(u,v) is spanned by two partials of S:

$$\frac{\partial S(u,v)}{\partial u} \quad \frac{\partial S(u,v)}{\partial v}$$

Normal ***to surface***

$$\vec{n} = \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}$$



- perpendicular to tangent plane

Any vector in tangent plane is tangential to S(u,v)

Curvature

Normal curvature ***of surface is defined for each tangential direction***

Principal curvatures **Kmin & Kmax**: ***maximum and minimum of normal curvature***

- Correspond to two **orthogonal** tangent directions
  - *Principal directions*
- Not necessarily partial derivative directions
- Independent of parameterization

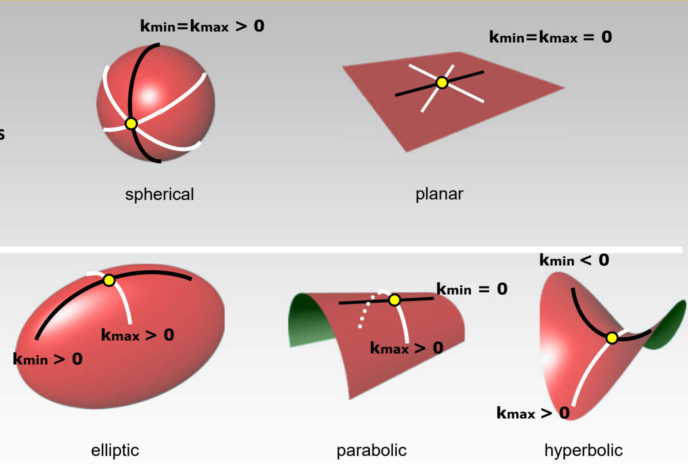
3D Curvature

Isotropic

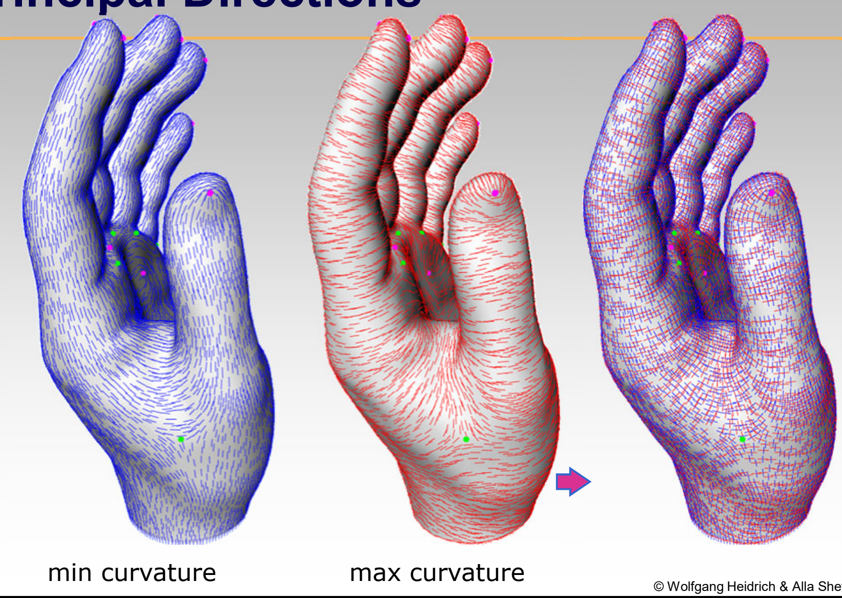
Equal in all directions

Anisotropic

2 distinct principal directions



Principal Directions



Curvature

Typical measures:

- ***Gaussian*** curvature

$$K = k_{\min} k_{\max}$$

- ***Mean*** curvature

$$H = \frac{k_{\min} + k_{\max}}{2}$$

Clicker questions:

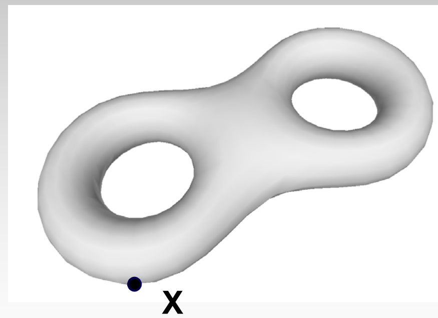
Which type of surface locally is point X?

A. Parabolic

B. Hyperbolic

C. Elliptic (non-isotropic)

D. Isotropic



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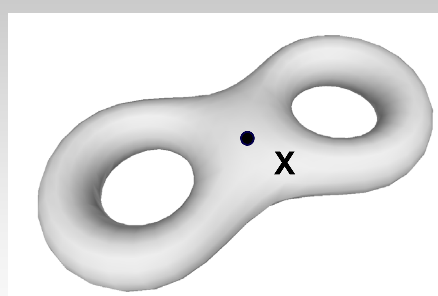

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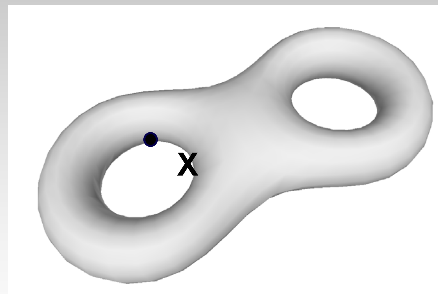
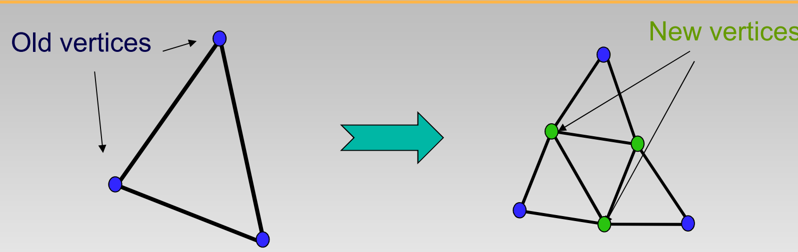


Triangular subdivision



Each face replaced by 4 new faces

Two kinds of new vertices:

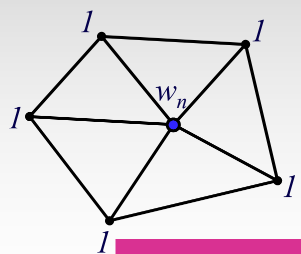
- Green vertices are associated with old edges
- Blue vertices are associated with old vertices

Loop's scheme

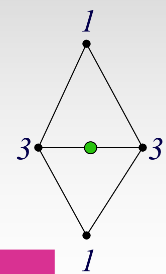
New vertex is weighted average of old vertices

List of weights called subdivision <u>mask</u> or <u>stencil</u>

- Rule for new blue vertices (n – vertex valence)



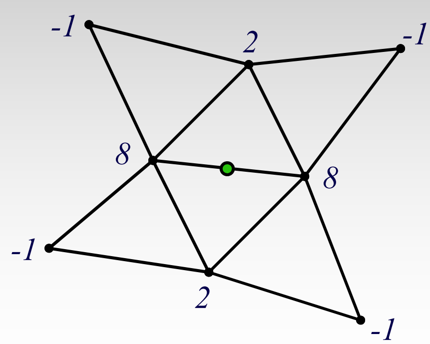
- Rule for new green vertices



$$w_n = \frac{64n}{40 - \left(3 + 2\cos(2\pi/n)\right)^2} - n$$

Butterfly Scheme

- Interpolatory scheme
- New blue vertices inherit location of old vertices
- New green vertices calculated by following stencil:



Subdivisons