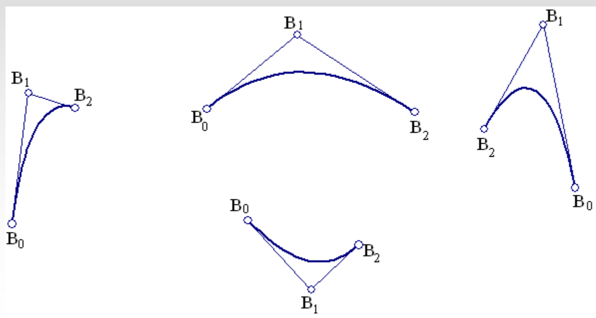
CPSC 424
Bézier Curves



Syllabus

Curves in 2D and 3D

- Implicit vs. Explicit vs. **Parametric** curves
- **Bézier curves,** de Casteljau algorithm
- Continuity
- B-Splines
- Subdivision Curves

Properties of Curves and Surfaces

Surfaces/Meshes/Advanced Topics

Clicker Test

Do you have a clicker?

A. Hardware
B. Mobile
C. No

Curves & Surfaces as Parametric Functions

Concept:

- Curve as function of artificial "time" parameter t

2D curve:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F_x(t) \\ F_y(t) \end{pmatrix} =: F(t); F : \mathcal{R} \mapsto \mathcal{R}^2$$

3D curve:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} F_x(t) \\ F_y(t) \\ F_z(t) \end{pmatrix} =: F(t); F : \mathcal{R} \mapsto \mathcal{R}^3$$

Parametric Curves

Advantage:

- Arbitrary curves in arbitrary dimensions

Still a problem:

- Unintuitive
  - *Try to find a formula for a specific curve you have in mind!*
- Hard to program with
  - *Deal with arbitrary mathematical functions*

Solution:

- Restrict yourself to specific class of functions

Spline Curves

Description = basis functions + coefficients

$$F(t) = \sum_{i=0}^{n} P_i B_i(t) = (x(t), y(t))$$

$$x(t) = \sum_{i=0}^{n} P_i^x B_i(t)$$

$$y(t) = \sum_{i=0}^{n} P_i^y B_i(t)$$

- Same basis functions for all coordinates

Polynomial Curves

Polynomial Curves:

- Restrict to polynomial functions of degree ≤ m:

$$\mathbf{x} = \sum_{i=0}^{m} \mathbf{b}_i t^i$$

- Note: $\mathbf{b}_i$ are vectors!
- Example curve in 2D:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \sum_{i=0}^{m} \begin{pmatrix} b_{x,i} \\ b_{y,i} \end{pmatrix} t^i$$

Polynomial Curves

Advantages:

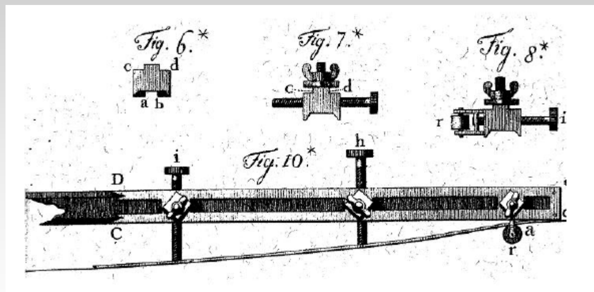
- Computationally easy to handle
  - *$\mathbf{b}_0 \ldots \mathbf{b}_m$ uniquely describe curve (finite storage, easy to represent)*

Disadvantages:

- Not all shapes representable
  - *Partially fix with piecewise functions later (splines)*
- Still not very intuitive
  - *Fix: represent polynomials in different basis*

Assign GEOMETRIC meaning to coefficients (base)

- Approximate/interpolate set of positions, derivatives, etc..



Parametric Curves

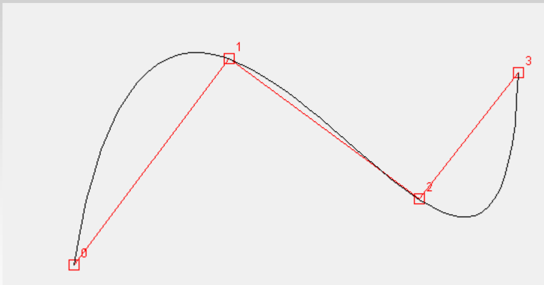
Commonly used classes:

- Polynomials
  - *Bézier curves, Hermite interpolation etc.*
- Piecewise polynomials
  - *B-splines*
- Rational and piecewise-rational curves
  - *Rational Bézier curves, rational B-splines (NURBS)*

Interpolate "Control" Points: Lagrange Polynomials

Use points we want to interpolate as controls

- Polynomial degree = number of input points



- https://www.ibiblio.org/e-notes/Splines/lagrange.html

Basis Functions: Lagrange Polynomials

- Given: m+1 parameter values $t_0 \ldots t_m$
- Define

$$L_i^m(t) := \prod_{j=0..m, j \neq i} \frac{t - t_j}{t_i - t_j}; i = 0 \ldots m$$

- Clear from definition:
  - *All L_i^m are polynomials of degree m*
  - $$L_i^m(t_k) = \begin{cases} 1; i = k \\ 0; else \end{cases}$$
  - *In particular, all L_i^m are linearly independent!*

Lagrangr Polynomials (cont)

- L_i^m are **linearly independent** & there are m+1 of them - basis for polynomials of degree up to m
- Can write any polynomial of degree up to m as

$$F(t) = \sum_{i=0}^{m} L_i^m(t_j) \cdot b_i$$

- In addition, we have for all *i*: $F(t_i) = b_i$
  - *In other words, the polynomial interpolates the points (t_i, b_i)*

Clicker Question

For a Lagrange curve with 4 control points positioned along a horizontal line. If I move the first point up, will the curve between two last points

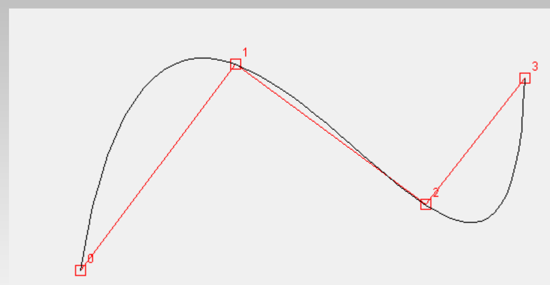
A. Move up

B. Move down

C. Stay where it was

D. No idea

Lagrange Polynomials



- https://www.ibiblio.org/e-notes/Splines/lagrange.html
- ***Oscillates unpredictably ☹***

Other Option: Hermite Curves

Other Option: Hermite Cubic Basis

Geometrically-oriented coefficients

- 2 positions + 2 tangents

Require $F(0)=P_0, F(1) = P_1, F'(0)=T_0, F'(1)=T_1$

Define basis function per requirement

$$F(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t)$$

Hermite Basis Functions

$$F(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t)$$

To enforce $C(0)=P_0, C(1) = P_1, C'(0)=T_0, C'(1)=T_1$ ***basis should satisfy***

$$h_{ij}(t): i, j = 0,1, t \in [0,1]$$

curve	$F(0)$	$F(1)$	$F'(0)$	$F'(1)$
$h_{00}(t)$	1	0	0	0
$h_{01}(t)$	0	1	0	0
$h_{10}(t)$	0	0	1	0
$h_{11}(t)$	0	0	0	1

Hermite Cubic Basis

Can satisfy with cubic polynomials as basis

$$h_{ij}(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

Obtain - solve 4 linear equations in 4 unknowns for each basis function

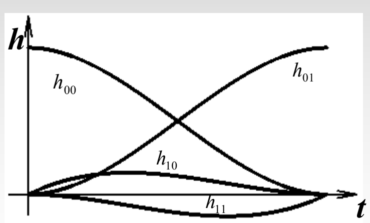
$$h_{ij}(t): i, j = 0,1, t \in [0,1]$$

curve	$F(0)$	$F(1)$	$F'(0)$	$F'(1)$
$h_{00}(t)$	1	0	0	0
$h_{01}(t)$	0	1	0	0
$h_{10}(t)$	0	0	1	0
$h_{11}(t)$	0	0	0	1

Hermite Cubic Basis

Four polynomials that satisfy the conditions

$$h_{00}(t) = t^2(2t-3)+1 \qquad h_{01}(t) = -t^2(2t-3)$$
$$h_{10}(t) = t(t-1)^2 \qquad h_{11}(t) = t^2(t-1)$$



https://codepen.io/liorda/pen/KrvBwr

Bézier Curves

Definition:

- Bézier curve is a polynomial curve that uses **Bernstein polynomials** as basis

$$F(t) = \sum_{i=0}^{m} \mathbf{b}_i B_i^m(t)$$

- b_i are called control points of Bézier curve
- Control polygon obtained by connecting control points with line segments

Bernstein Polynomials

$$B_i^m(t) := \binom{m}{i} t^i (1-t)^{m-i}; i = 0..m; t \in [0,1],$$

$$\binom{m}{i} = \frac{m!}{(m-i)!i!}$$

Clicker Question

$$B_i^m(t) := \binom{m}{i} t^i (1-t)^{m-i}; i = 0..m; t \in [0,1],$$

$$\binom{m}{i} = \frac{m!}{(m-i)!\,i!}$$

What is the value of B_0^m at t=0?

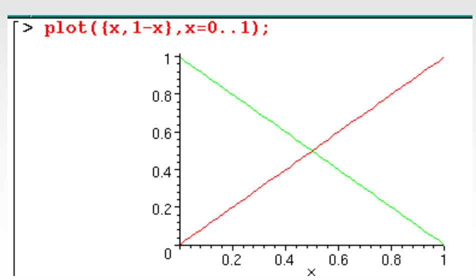
A. Depends on m
B. 1
C. 0

Bernstein Polynomials

$$B_i^m(t) := \binom{m}{i} t^i (1-t)^{m-i}; i = 0..m; t \in [0,1],$$

$$\binom{m}{i} = \frac{m!}{(m-i)!\,i!}$$

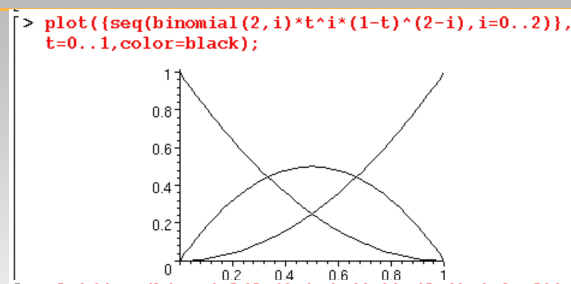
- Graph for degree m=1:



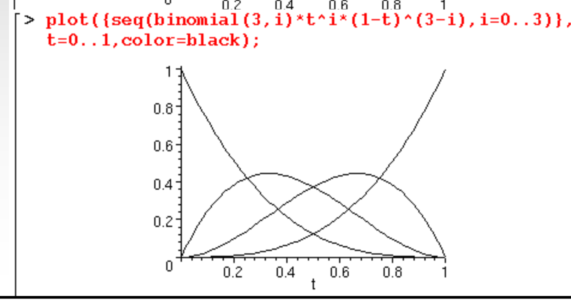
Q: Which color is B_0^1 ?
A. Red
B. Green

Bernstein Polynomials

- Graph for m=2:



- Graph for m=3:



Bernstein Polynomials

$$B_i^m(t) := \binom{m}{i} t^i (1-t)^{m-i}; i = 0..m; t \in [0,1]$$

Properties:

- $B_i^m(t)$ is a polynomial of degree m
- $B_i^m(t) \geq 0 \text{ for } t \in [0,1]; B_0^m(0) = 1; B_i^m(0) = 0 \text{ for } i \neq 0$
- $B_i^m(t) = B_{m-i}^m(1-t)$

Bernstein Polynomials

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$$B_i^m(t) := \binom{m}{i} t^i (1-t)^{m-i}; i = 0..m; t \in [0,1]$$

Properties:

- $B_i^m(t)$ has exactly one maximum in the interval 0..1. It is at t=i/m (proof: compute derivative…)
- W/o proof: all (m+1) functions B_i^m are linearly independent
  - *Thus they form a basis for all polynomials of degree $\leq m$*

Bernstein Polynomials

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More properties

- $\sum_{i=0}^{m} B_i^m(t) = (t + (1-t))^m \equiv 1$
  - *(proof: apply Binomial Theorem to definition)*
- $B_i^m(t) = t \cdot B_{i-1}^{m-1}(t) + (1-t) \cdot B_i^{m-1}(t)$
  - *(proof on board)*
- Important (later) for fast evaluation algorithm of Bézier curves (de Casteljau algorithm)

Properties of Bézier Curves (Pierre Bézier, Renault, about 1970)

Easy to see:

- Endpoints b_0 and b_m of control polygon interpolated & corresponding parameter values are t=0 and t=1

Without proof for the moment (will be easier to show later):

- Bézier curve is tangential to control polygon at endpoints
- Curve lies within convex hull of control points
- Curve is *affine invariant*
- There is a fast, recursive evaluation algorithm

Clicker Question

Is a Bezier curve defined by colinear points a straight line?

A. Always

B. Never

C. Sometimes