CPSC 424
Subdivision

Syllabus

Curves in 2D and 3D

Properties of Curves and Surfaces

Surfaces

Polygonal meshes

- Subdivision

Subdivision schemes for surfaces

Each iteration

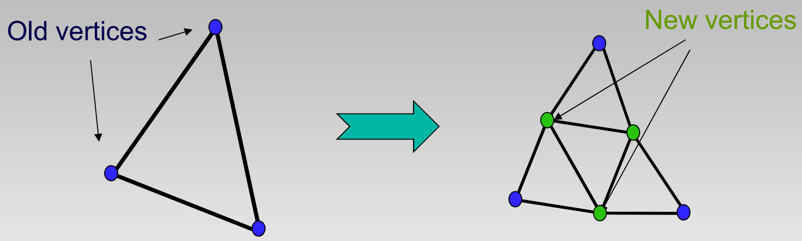
- Refine *control net* (mesh) - increase number of vertices (approximately) * 4

Every subdivision method has:

- Method to generate net topology
- Rules to calculate location of new/old vertices

Mesh vertices converge to limit surface

Triangular subdivision



Each face replaced by 4 new faces

Two kinds of new vertices:

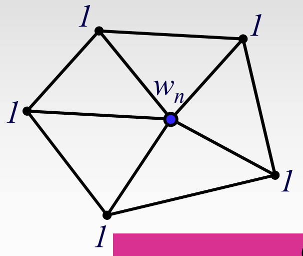
- Green vertices are associated with old edges
- Blue vertices are associated with old vertices

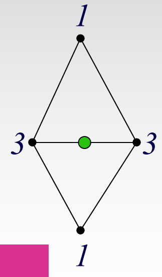
Loop's scheme

New vertex is weighted average of old vertices

List of weights called subdivision <u>mask</u> or <u>stencil</u>

- Rule for new blue vertices (*n* – vertex valence)
- Rule for new green vertices

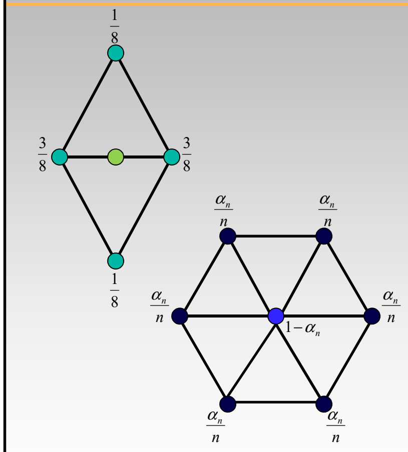


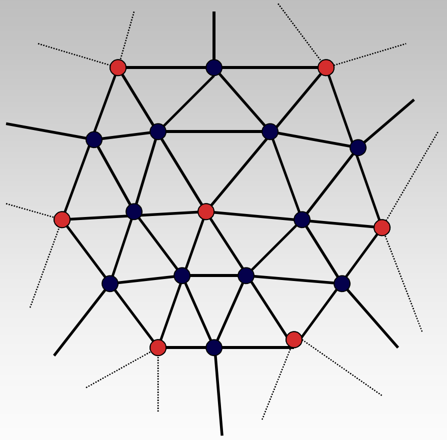


$$w_n = \frac{64n}{40 - (3 + 2\cos(2\pi/n))^2} - n$$

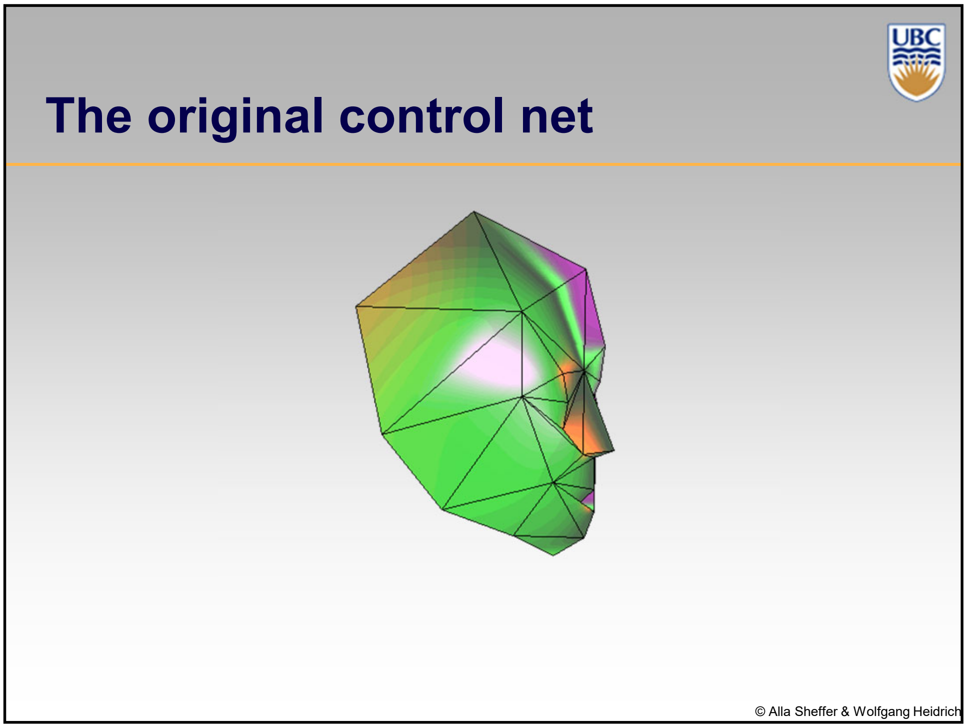
Loop Subdivision Masks

(from J.Hart)





$$\alpha_n = \frac{1}{64}\left(40 - \left(3 + 2\cos\left(\frac{2\pi}{n}\right)\right)^2\right)$$



The original control net

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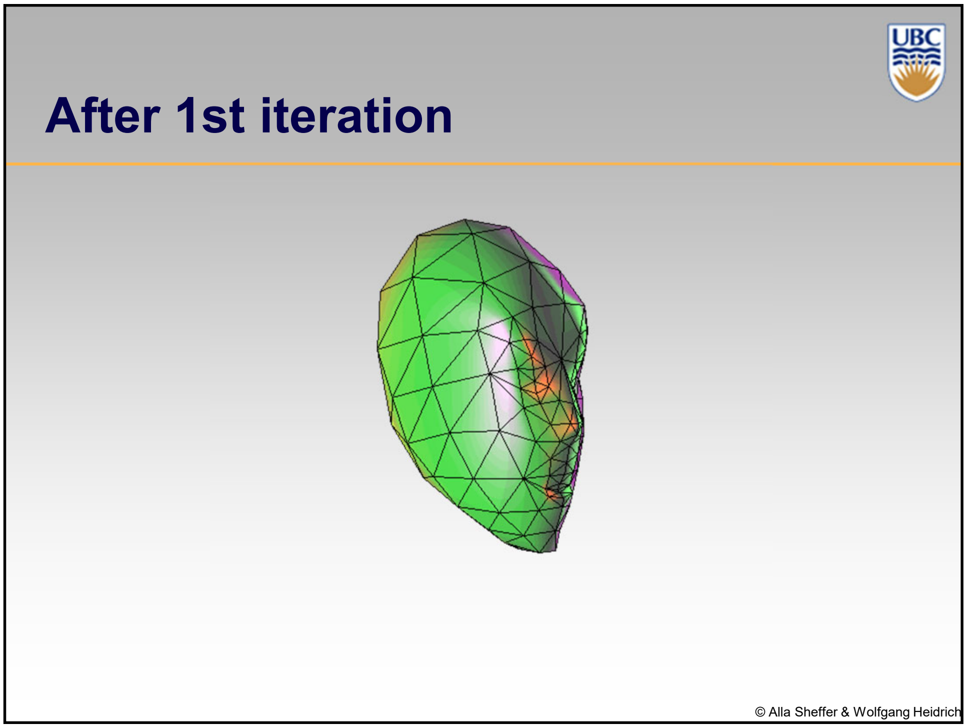

After 1st iteration

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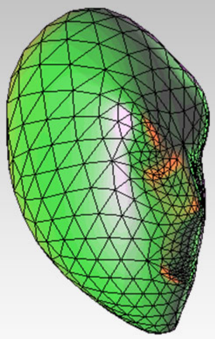
After 2nd iteration



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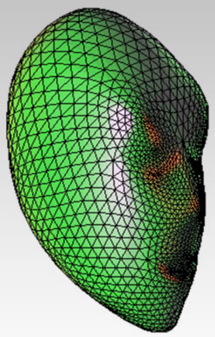
After 3rd iteration



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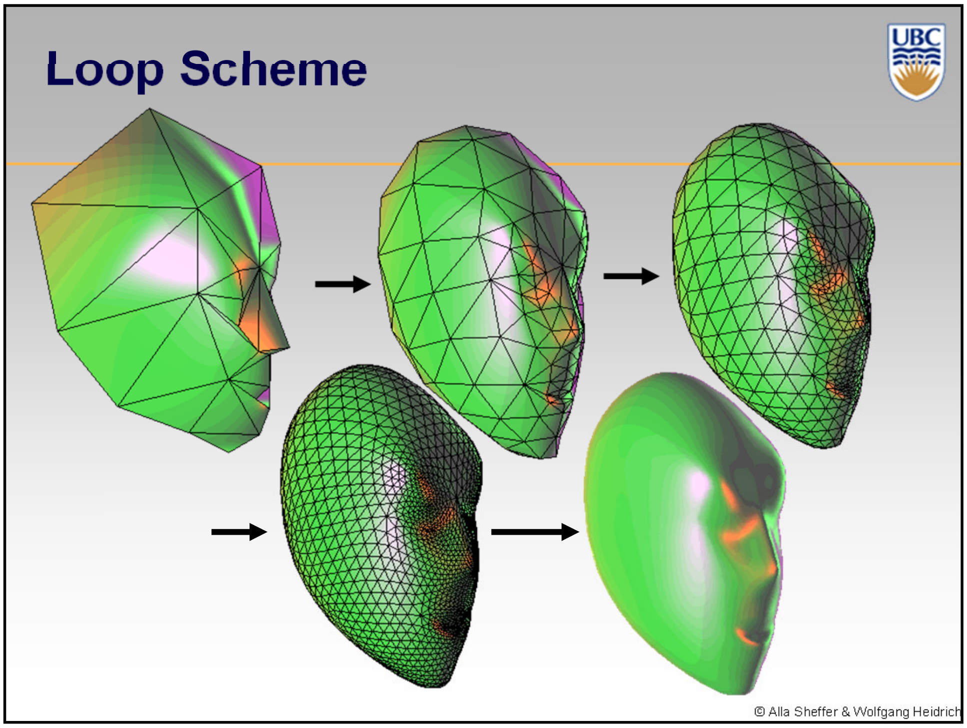
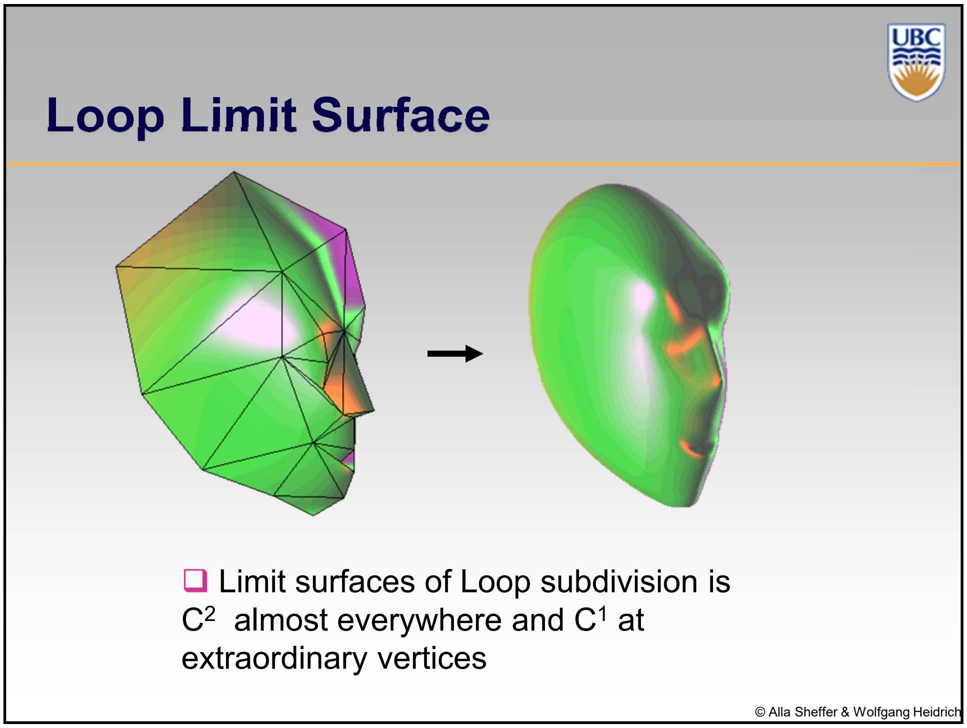
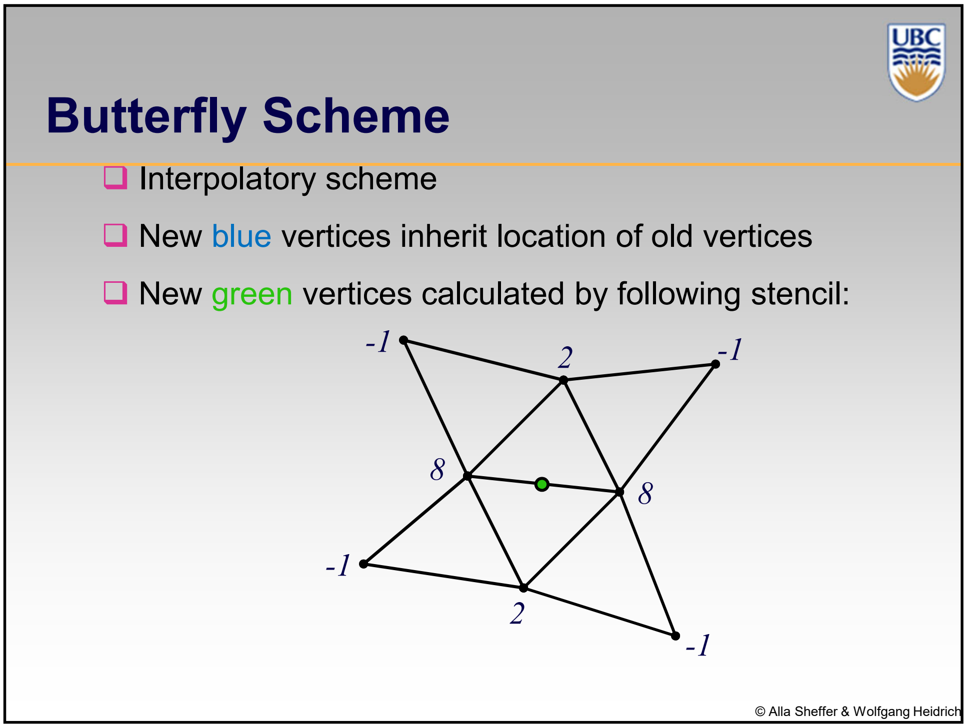


Loop Scheme



Loop Limit Surface

- Limit surfaces of Loop subdivision is C^2 almost everywhere and C^1 at extraordinary vertices



Butterfly Scheme

- ☐ Interpolatory scheme
- ☐ New blue vertices inherit location of old vertices
- ☐ New green vertices calculated by following stencil:

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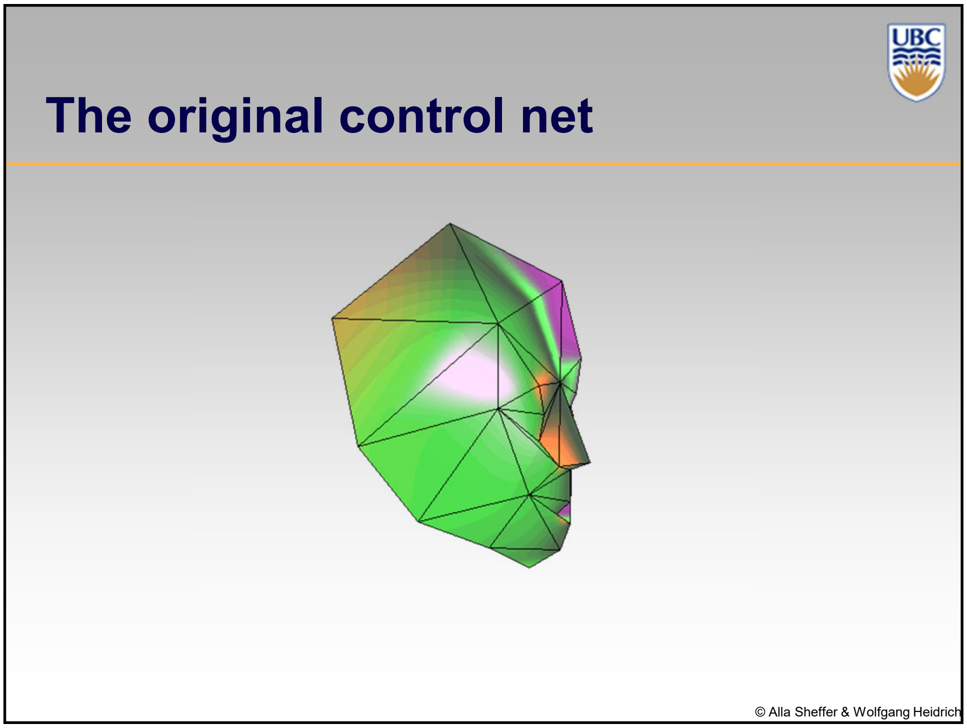

The original control net

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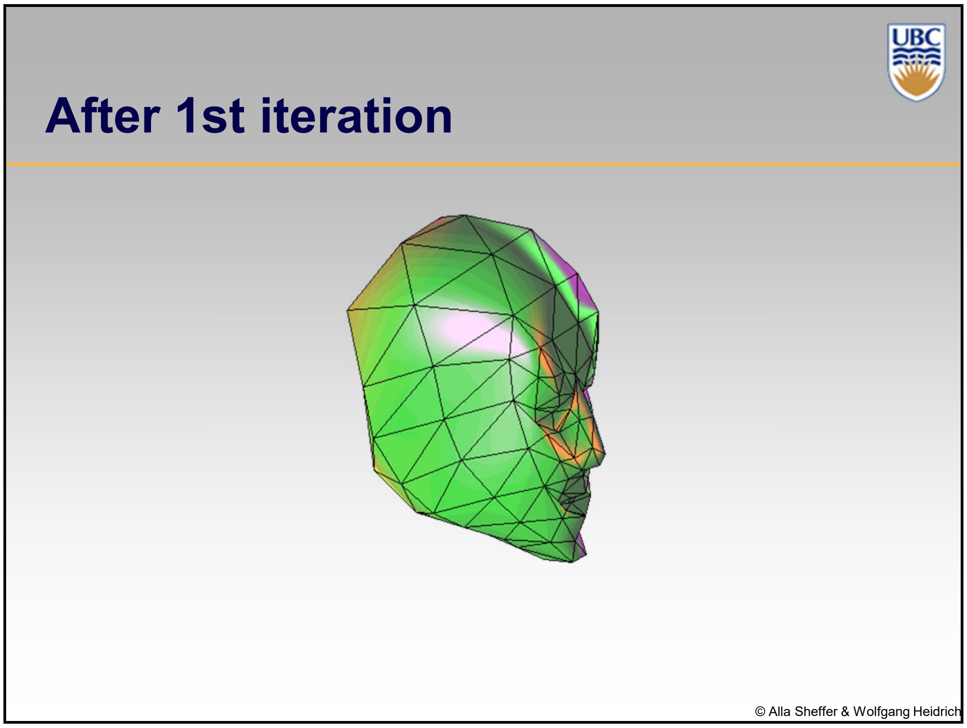
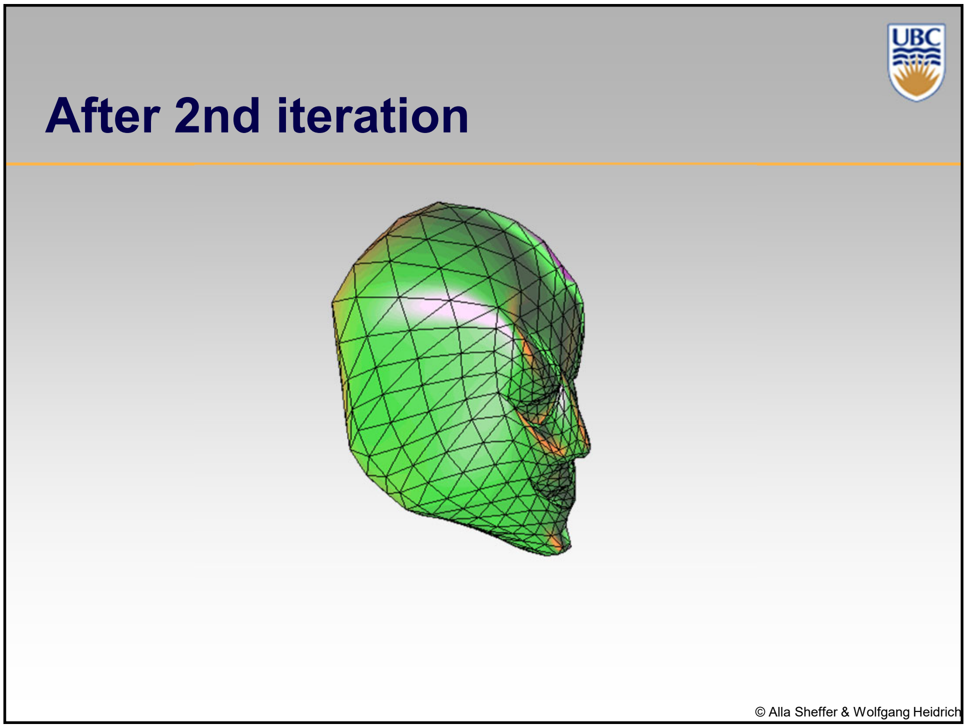
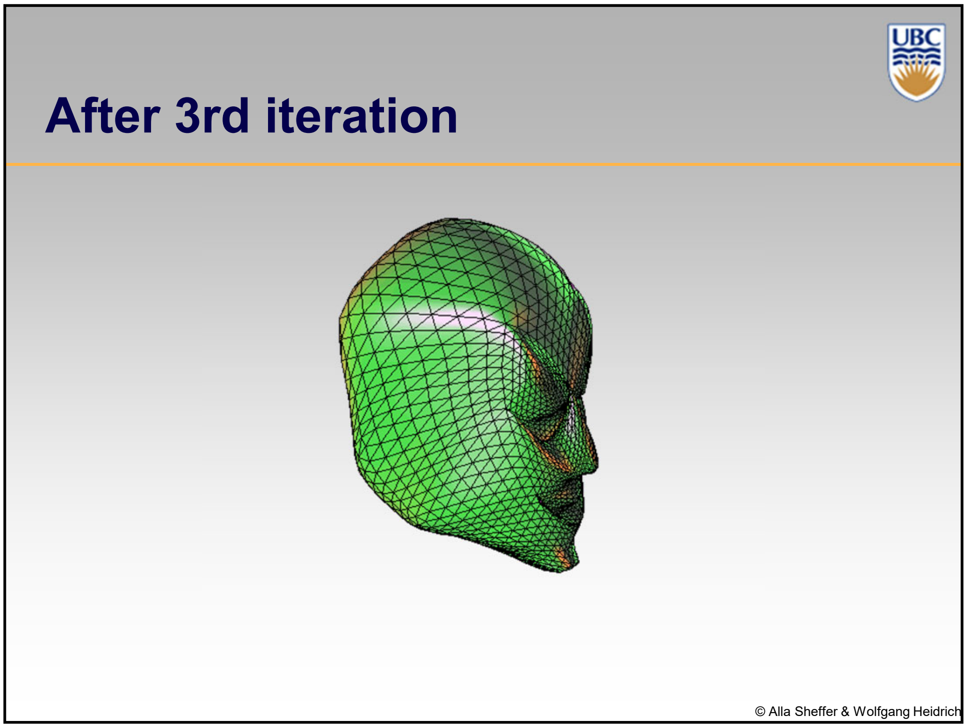


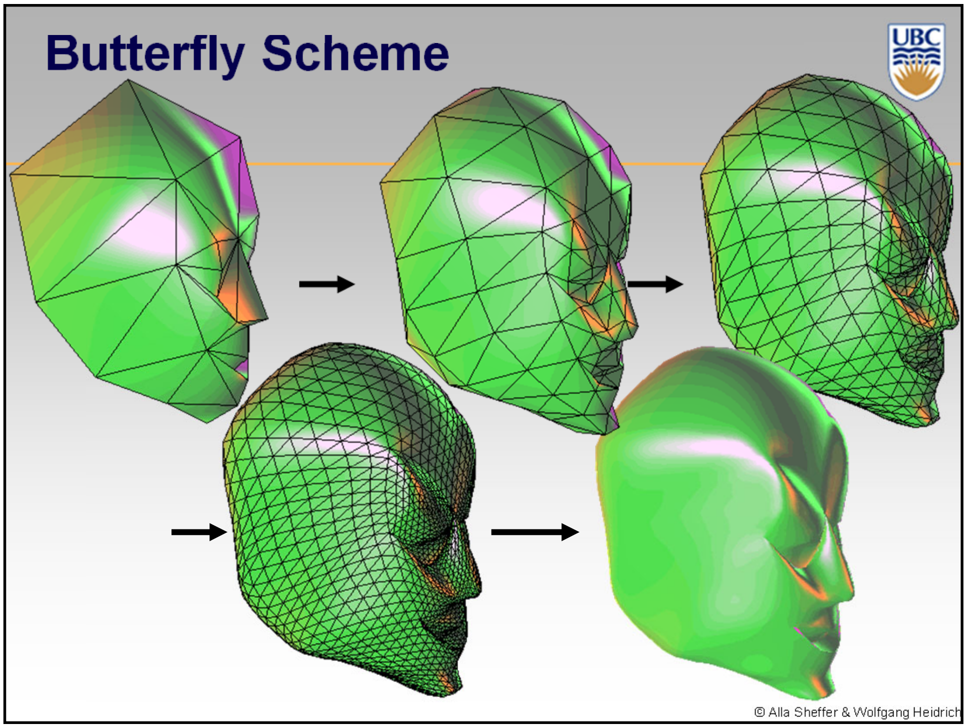
After 1st iteration



After 2nd iteration

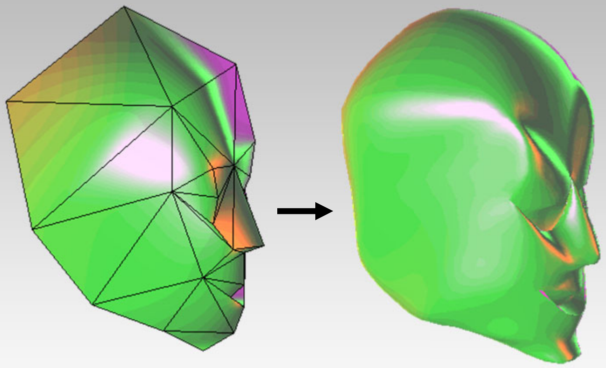


After 3rd iteration



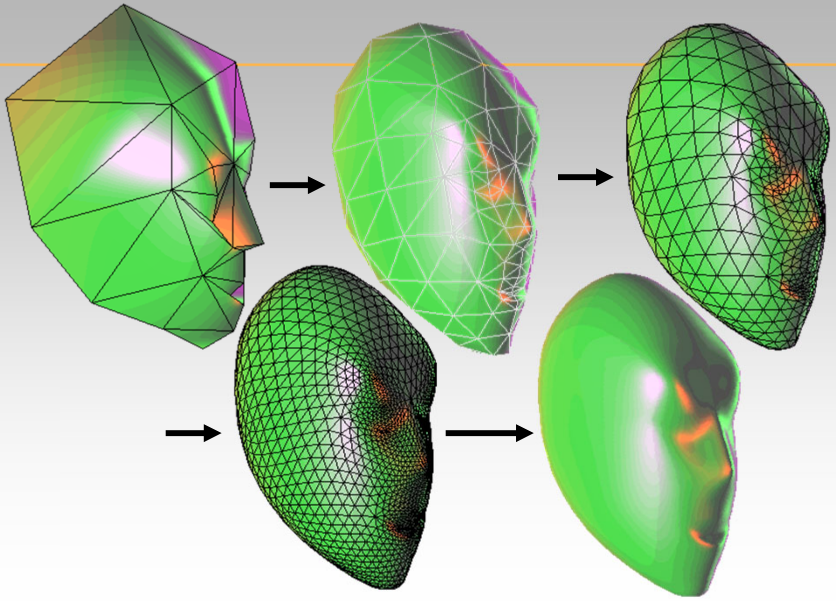
Butterfly Scheme

Butterfly limit surface

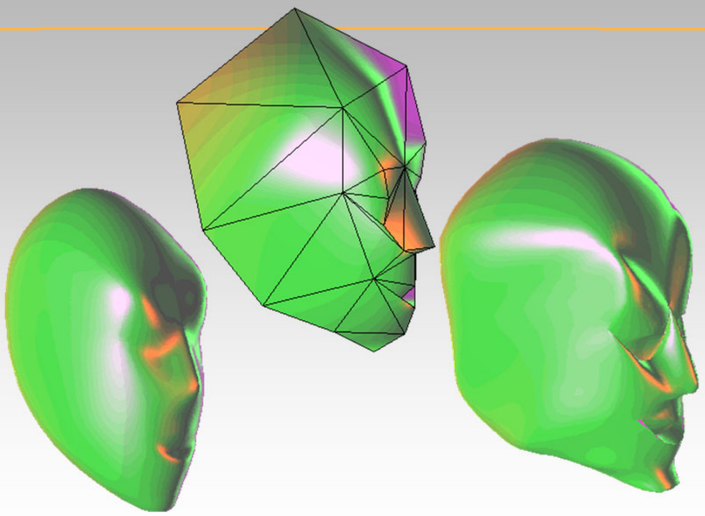


- Limit surfaces of Butterfly subdivision are C^1 almost everywhere except at extraordinary vertices but do not have second derivative
  - Modified version is C^1 everywhere

Loop Scheme



Subdivisons



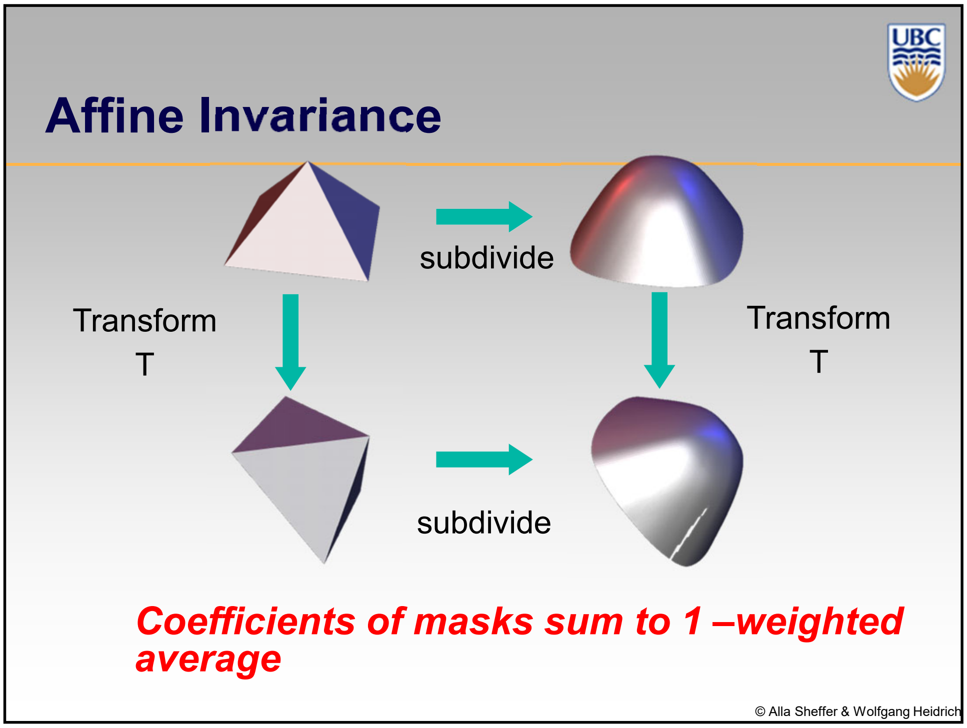
Scheme Zoo

More schemes:

- Catmul-Clark
- Kobbelt
- Duo-Sabin
- …

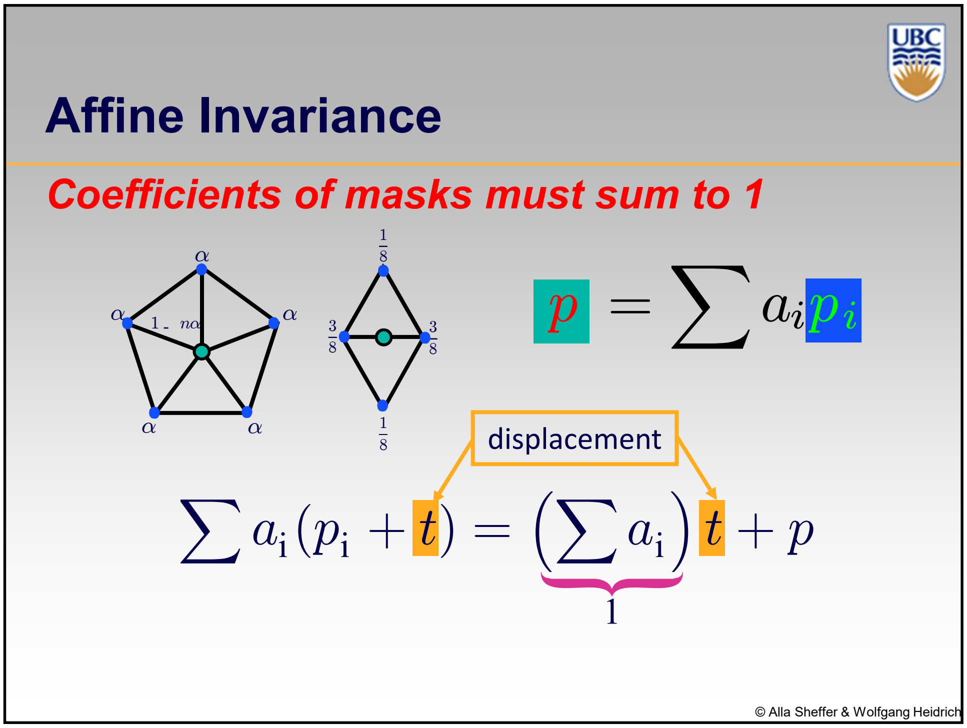
Proving scheme works:

- **Convergence**
- Degree of continuity
- Affine invariance



Affine Invariance

Coefficients of masks sum to 1 –weighted average



Affine Invariance

Coefficients of masks must sum to 1

$$p = \sum a_i p_i$$

displacement

$$\sum a_i (p_i + t) = \underbrace{\left(\sum a_i\right)}_{1} t + p$$

Analysis of Subdivision

Test:

- Convergence
- Smoothness

Goals:

- Help to choose the rules
- Ensure that all surfaces have desired properties

Plan

- Define subdivision surfaces
- Relate properties to coefficients

Subdivision Matrix

Relate control on finer level to coarser level

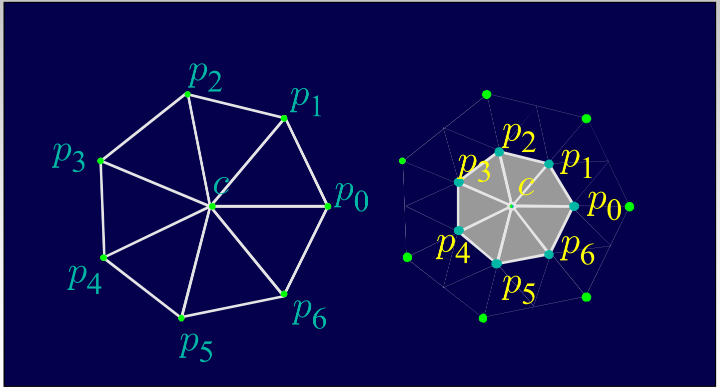
Useful for

- Analysis of properties
  - *smoothness*
  - *affine invariance*
- Formulas for normals

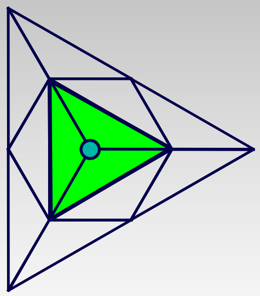
- Explicit evaluation of surfaces at arbitrary points of the domain

Controls of K-Sided Patch

Simplest case - consider only 1-ring



Subdivision Matrix



$$\left(\begin{array}{c|ccc|ccc|ccc}
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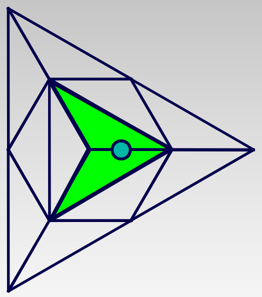
\hline
3/8 & 3/8 & 1/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/8 & 1/8 & 3/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/8 & 1/8 & 1/8 & 3/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1/16 & 5/8 & 1/16 & 1/16 & 1/16 & 0 & 0 & 1/16 & 0 & 1/16 \\
1/16 & 1/16 & 5/8 & 1/16 & 0 & 1/16 & 0 & 1/16 & 1/16 & 0 \\
1/16 & 1/16 & 1/16 & 5/8 & 0 & 0 & 1/16 & 0 & 1/16 & 1/16 \\
\hline
1/8 & 3/8 & 3/8 & 0 & 0 & 0 & 0 & 1/8 & 0 & 0 \\
1/8 & 0 & 3/8 & 3/8 & 0 & 0 & 0 & 0 & 1/8 & 0 \\
1/8 & 3/8 & 0 & 3/8 & 0 & 0 & 0 & 0 & 0 & 1/8
\end{array}\right)$$

Subdivision Matrix



$$\left(\begin{array}{c|ccc|ccc|ccc}
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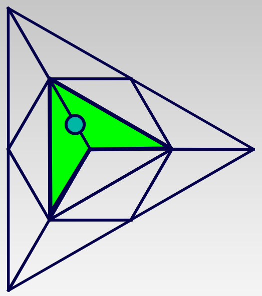
3/8 & 3/8 & 1/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/8 & 1/8 & 3/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/8 & 1/8 & 1/8 & 3/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1/16 & 5/8 & 1/16 & 1/16 & 1/16 & 0 & 0 & 1/16 & 0 & 1/16 \\
1/16 & 1/16 & 5/8 & 1/16 & 0 & 1/16 & 0 & 1/16 & 1/16 & 0 \\
1/16 & 1/16 & 1/16 & 5/8 & 0 & 0 & 1/16 & 0 & 1/16 & 1/16 \\ \hline
1/8 & 3/8 & 3/8 & 0 & 0 & 0 & 0 & 1/8 & 0 & 0 \\
1/8 & 0 & 3/8 & 3/8 & 0 & 0 & 0 & 0 & 1/8 & 0 \\
1/8 & 3/8 & 0 & 3/8 & 0 & 0 & 0 & 0 & 0 & 1/8
\end{array}\right)$$

Subdivision Matrix



$$\left(\begin{array}{c|ccc|ccc|ccc}
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3/8 & 1/8 & 3/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/8 & 1/8 & 1/8 & 3/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1/16 & 5/8 & 1/16 & 1/16 & 1/16 & 0 & 0 & 1/16 & 0 & 1/16 \\
1/16 & 1/16 & 5/8 & 1/16 & 0 & 1/16 & 0 & 1/16 & 1/16 & 0 \\
1/16 & 1/16 & 1/16 & 5/8 & 0 & 0 & 1/16 & 0 & 1/16 & 1/16 \\ \hline
1/8 & 3/8 & 3/8 & 0 & 0 & 0 & 0 & 1/8 & 0 & 0 \\
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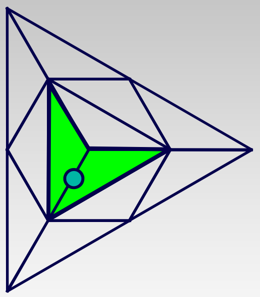
\end{array}\right)$$

Subdivision Matrix



$$\left(\begin{array}{c|ccc|ccc|ccc}
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3/8 & 1/8 & 1/8 & 3/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1/16 & 5/8 & 1/16 & 1/16 & 1/16 & 0 & 0 & 1/16 & 0 & 1/16 \\
1/16 & 1/16 & 5/8 & 1/16 & 0 & 1/16 & 0 & 1/16 & 1/16 & 0 \\
1/16 & 1/16 & 1/16 & 5/8 & 0 & 0 & 1/16 & 0 & 1/16 & 1/16 \\ \hline
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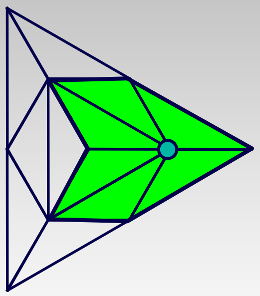
\end{array}\right)$$

Subdivision Matrix



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3/8 & 1/8 & 1/8 & 3/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
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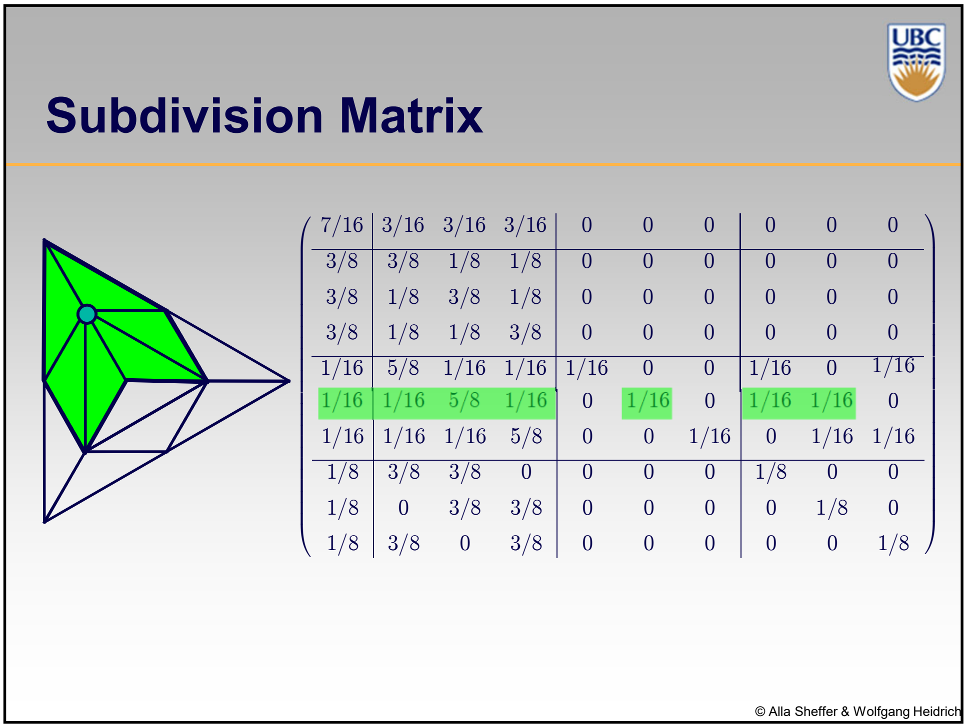

Subdivision Matrix

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3/8 & 1/8 & 1/8 & 3/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
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1/16 & 1/16 & 5/8 & 1/16 & 0 & 1/16 & 0 & 1/16 & 1/16 & 0 \\
1/16 & 1/16 & 1/16 & 5/8 & 0 & 0 & 1/16 & 0 & 1/16 & 1/16 \\
\hline
1/8 & 3/8 & 3/8 & 0 & 0 & 0 & 0 & 1/8 & 0 & 0 \\
1/8 & 0 & 3/8 & 3/8 & 0 & 0 & 0 & 0 & 1/8 & 0 \\
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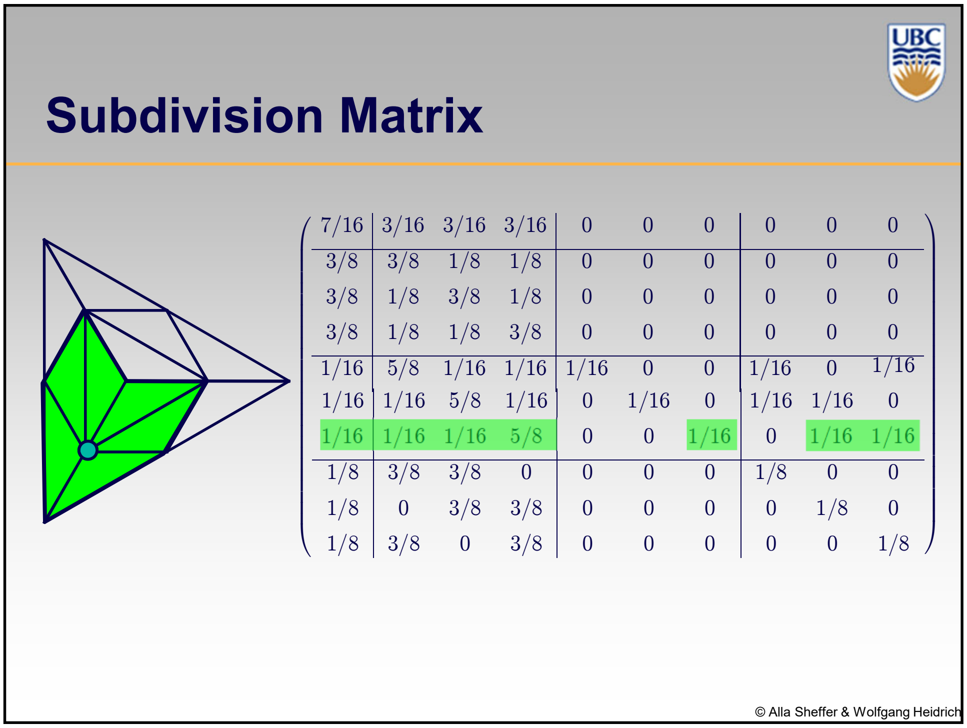
\end{array}\right)$$



Subdivision Matrix

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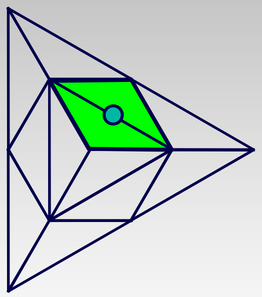
Subdivision Matrix



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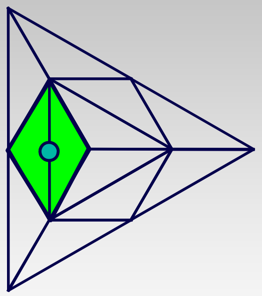
\end{array}\right)$$

Subdivision Matrix



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3/8 & 1/8 & 3/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
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1/16 & 1/16 & 5/8 & 1/16 & 0 & 1/16 & 0 & 1/16 & 1/16 & 0 \\
1/16 & 1/16 & 1/16 & 5/8 & 0 & 0 & 1/16 & 0 & 1/16 & 1/16 \\ \hline
1/8 & 3/8 & 3/8 & 0 & 0 & 0 & 0 & 1/8 & 0 & 0 \\
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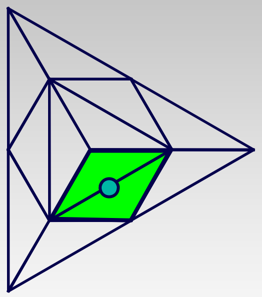
1/8 & 3/8 & 0 & 3/8 & 0 & 0 & 0 & 0 & 0 & 1/8
\end{array}\right)$$

Subdivision Matrix



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3/8 & 1/8 & 3/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/8 & 1/8 & 1/8 & 3/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1/16 & 5/8 & 1/16 & 1/16 & 1/16 & 0 & 0 & 1/16 & 0 & 1/16 \\
1/16 & 1/16 & 5/8 & 1/16 & 0 & 1/16 & 0 & 1/16 & 1/16 & 0 \\
1/16 & 1/16 & 1/16 & 5/8 & 0 & 0 & 1/16 & 0 & 1/16 & 1/16 \\ \hline
1/8 & 3/8 & 3/8 & 0 & 0 & 0 & 0 & 1/8 & 0 & 0 \\
1/8 & 0 & 3/8 & 3/8 & 0 & 0 & 0 & 0 & 1/8 & 0 \\
1/8 & 3/8 & 0 & 3/8 & 0 & 0 & 0 & 0 & 0 & 1/8
\end{array}\right)$$

Eigen Decomposition

Diagonalize subdivision matrix

- eigenvectors $x_i,\ i = 0..N$
- eigenvalues $\lambda_i,\ i = 0..N$
- p: vector of points in a neighborhood

$$p = \sum_i a_i x_i$$

(N+1)-vector of 3D points

3D vector

Eigenvectors

"Good" case:

- $\lambda_0 = 1$ & $|\lambda_i| < 1, i = 1, \ldots, n-1$

$$S^m p = a_0 x_0 + \lambda_1^m a_1 x_1 + \lambda_2^m a_2 x_2 + \lambda_3^m a_3 x_3 + \ldots$$

limit position

tangent vectors

- can make a_0 zero by moving control points (by affine invariance)

Subdominant Eigenvectors

Next higher order terms

- assume $\lambda = \lambda_1 = \lambda_2 > |\lambda_3|$
- move control points so that $a_0 = 0$

$$\frac{1}{\lambda^m} S^m p = (a_1 x_1 + a_2 x_2) + \left(\frac{\lambda_3}{\lambda}\right)^m x_3 + \ldots$$

subdominant eigenvectors

vanishes