Curves: Review

Curves in 2D and 3D

- Implicit vs. Explicit vs. Parametric curves
- Bézier curves, de Casteljau algorithm
- Continuity
- B-Splines
- Subdivision Curves
**Bézier Curves**

**Definition:**
- Bézier curve is a polynomial curve that uses **Bernstein polynomials** as basis
- \( F(t) = \sum_{i=0}^{m} b_i B_i^m(t) \)
- \( b_i \) are called **control points** of Bézier curve
- Control polygon obtained by connecting control points with line segments

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**Bernstein Polynomials**

\[
B_i^m(t) := \binom{m}{i} t^i (1-t)^{m-i}; i = 0..m; t \in [0,1],
\]

\[
\binom{m}{i} = \frac{m!}{(m-i)!i!}
\]
Properties of Bézier Curves

- Endpoints $b_0$ and $b_m$ of control polygon interpolated & corresponding parameter values are $t=0$ and $t=1$
- Bézier curve is tangential to control polygon at endpoints
- Curve lies within convex hull of control points
- Curve is **affine invariant**
- There is a fast, recursive evaluation algorithm – de Casteljau algorithm

De Casteljau Algorithm

**Graphical Interpretation:**

- Determine point $F(1/2)$ for the cubic Bézier curve given by the following four points:
**Observation**

*De Casteljau generates 2 new control polygons!*

- For parameter interval $[0, 1/2]$, and $[1/2, 1]$

- Can be used to recursively subdivide control polygon

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**Bezier Subdivision**

*Cubic case:*

![Diagram showing Bezier subdivision process](image)
**Degree Elevation**

*Replace degree m polynomial (m+1 control points) with degree m+1 polynomial (m+2 control points):*

- New control points $b'_i$:

$$b'_i = \frac{1}{m+1} \left[ i \cdot b_{i-1} + (m+1-i) b_i \right]$$

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**Degree Elevation**

*Examples:*

- $m=2$
- $m=3$
Derivatives of Bézier Curves

**Theorem (proof on board):**
- The derivative of a Bézier curve

\[
F(t) := \sum_{i=0}^{m} B_i^m(t) \cdot b_i
\]

is given as

\[
F'(t) := m \cdot \sum_{i=0}^{m-1} B_i^{m-1}(t) \cdot (b_{i+1} - b_i)
\]

Continuity

**Def:**
- A curve \( F(t) \) is called \( C^k \)-continuous if its \( k \)th derivative \( F^{(k)}(t) \) exists (i.e. is continuous) everywhere

**Note:**
- Polynomial curves are infinitely continuous

**Def:**
- Two curve segments \( F(t) \) and \( G(t) \) are called \( C^k \)-continuous at \( t_0 \) if their first \( k \) derivatives match at \( t_0 \)
**Bezier Continuity**

- **C^0**: share end control points \( b_m = b'_0 \)
- **C^1**: \( b_m - b_{m-1} = b'_1 - b'_0 \)
- **C^2**: \( b_m - b_{m-1} \) collinear to \( b'_1 - b'_0 \)

**B-Splines**

Idea: Generate basis where functions are continuous across domains

Control point controls set of basis functions (to preserve continuity)

Alternative view: continuous basis functions defined on several domains
Uniform Cubic B-Spline Curves

**Definition**

\[ C(t) = \sum_{i=0}^{n-1} P_i N_i^3(t) \quad t \in [3, n] \]

\[ N_i^3(t) = \begin{cases} 
  r^3 / 6 & r = t - i \quad t \in [i, i+1] \\
  (-3r^3 + 3r^2 + 3r + 1) / 6 & r = t - i - 1 \quad t \in [i+1, i+2] \\
  (3r^3 - 6r^2 + 4) / 6 & r = t - i - 2 \quad t \in [i+2, i+3] \\
  (1-r)^3 / 6 & r = t - i - 3 \quad t \in [i+3, i+4] 
\end{cases} \]

For any \( t \in [3, n] \)

\[ \sum_{i=j-3}^{j} N_i^3(t) = 1 \]

For any \( t \in [j, j+1] \) only 4 basis functions are non zero

\[ \sum_{i=j}^{j+3} N_i^3(t) = 1 \]

Any point on cubic B-Spline is affine combination of at most 4 control points.
Boundary Conditions for B-Splines

**B-Splines do not interpolate any control points**

- in particular end points

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**B-Splines**

**Direct recursion formula:**

\[
N_i^0(t) = \begin{cases} 
1 & ; u_i \leq t < u_{i+1} \\
0 & ; \text{else} 
\end{cases}
\]

\[
N_i^l(t) = \frac{t - u_i}{u_{i+l} - u_i} \cdot N_i^{l-1}(t) + \frac{u_{i+l+1} - t}{u_{i+l+1} - u_{i+1}} \cdot N_{i+1}^{l-1}(t)
\]

**Note:**
- Not an affine combination
NURBs

- B-Spline (B-spline basis)
- Non-Uniform – different interval lengths (knots)
- Rational – rational basis functions

\[ C(t) = \frac{\sum_{i=0}^{n-1} w_i P_i N_i^3(t)}{\sum_{i=0}^{n-1} w_i N_i^3(t)} \quad t \in [3, n] \]

Subdivision:
Corner Cutting – Chaikin Algorithm
Cubic B-Spline (corner cutting)

The 4-point scheme
Proving scheme works

Proving scheme works:
• Convergence
  – *Will do on board & more details later*
• Degree of continuity
• Affine invariance
  – *As long as weights sum to 1*
  – *Proof on board*

Subdivision Matrix

*Example: Chaikin subdivision*

\[
\begin{bmatrix}
P_0' \\
0 \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
1/4 & 3/4 & 0 & 0 \\
0 & 3/4 & 1/4 & 0 \\
0 & 1/4 & 3/4 & 0 \\
0 & 0 & 3/4 & 1/4
\end{bmatrix}
\begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3
\end{bmatrix}
\]

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Syllabus

Curves in 2D and 3D

• ...

• Subdivision Curves

Properties of Curves and Surfaces

• Differential Geometry:
  – arc length
  – curvature
  – Fresnet frame

Surfaces

Regularity

Definition:

• Differentiable parametric curve \( F(t) : [a, b] \rightarrow \mathbb{R}^3 \) is called regular if

\[
F'(t) \neq 0, \quad \forall t \in [a, b]
\]

• (I.e. if the tangent vector is not 0 anywhere)

Note:

• Bézier curves not necessarily regular…
**Equivalence/Reparameterization**

**Definition**

- Two regular curves

\[ F(t) : [a, b] \alpha \mathbb{R}^3 \quad G(t) : [c, d] \alpha \mathbb{R}^3 \]

are geometrically equivalent \( F \cong G \) if there is a strictly monotonic, differentiable function

\[ \varphi(t) : [a, b] \alpha [c, d] \]

with

\[ F(t) = G(\varphi(t)) \]
Clicker Question

Are the curves geometrically equivalent?
F(t) = (t,t) t in [0,1] and G(t) = (t/3,t/3) t in [0,3]

- A. Yes
- B. No
- Not enough information

Arc Length

Definition
- Arc length of regular curve \( F(t):[a,b] \rightarrow \mathbb{R}^3 \) given as

\[
s(t) := \int_a^t \| F'(\tilde{t}) \| \, d\tilde{t}
\]

Parameterization by arc length
\( G(s) \) with \( G(s(t)) = F(t) \)

- Note: this is a **canonical** representation for any curve
- Point is traveling along \( G \) with constant speed 1
Curvature

Definition

- Let $G$ be a curve parameterized by arc length
- We introduce the following terms:
  - Unit tangent $T(s) := G'(s)$
  - Curvature vector $K(s) := G''(s)$
  - Curvature $\kappa(s) := \| K(s) \|
  - Principal normal $N(s) := K(s) / \kappa(s)$
  - Binormal $B(s) := T(s) \times N(s)$

Corresponds to radius of osculating circle $R = 1 / \kappa$

Measure curve bending
**Frenet Frame**

**Theorem:**
- Curvature vector and tangent vector are perpendicular:

\[ K(s) \perp T(s) \]

**Note:**
- Therefore, \( T, N, \) and \( B \) form an orthonormal coordinate frame
- This is called the Frenet Frame

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**Torsion**

*With the same argument we get*

\[ B'(s) = \tau(s) \cdot N(s) \]

**Note:**
- \( B' \) is the torsion vector
- \( \tau \) is the torsion, and indicates how much the curve twists out of the plane (\( \tau = 0 \) means perfectly planar)
**Fundamental Theorem of Curves**

**Theorem:**
- For given functions $\kappa(s)$, $\tau(s)$ there exists exactly one (except for rotations and translations) unique curve that is parameterized by arc length and has curvature $\kappa(s)$, and torsion $\tau(s)$

**Proof:**
- Quite complex, see for example
  - *Da Carmo*
    "Differential Geometry of Curves and Surfaces"

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**Geometric Continuity**

**Definition:**
- Two curves
  \[ F_1(t) : [a, b] \alpha \mathbb{R}^3, F_2(t) : [b, c] \alpha \mathbb{R}^3 \]
  are $G^k$-continuous (geometrically continuous of degree $k$), if there are reparameterizations
  \[ G_1(t) \cong F_1(t) \text{ and } G_2(t) \cong F_2(t) \]
  that are $C^k$ continuous, i.e.:
  \[ G_1^l(t) = G_2^l(t), l = 0 \text{ to } k \]
  at shared parameter interval endpoint
Basic surfaces

Extrusion

**Concept:**
- Move a curve ("profile") along a line segment
- The union of all points visited defines the surface

\[ S(u, v) = F(u) + \overrightarrow{P_1P_2}v \]
Surfaces of Revolution

Concept:
- Rotate profile curve around an axis
- \( R(v) \) rotation matrix (\( v \) in \([0, 2\pi]\))

\[
S(u, v) = R(v)F(u)
\]

Sweeping

Concept:
- Generalize extrusion & revolution - sweep along arbitrary curve
- To orient profile at any point
  - user specified
  - use Fresnet frame
Bilinear Patches

Bilinear interpolation of 4 3D points - 2D analog of 1D linear interpolation between 2 points in the plane

Given \( P_{00}, P_{01}, P_{10}, P_{11} \) - associated parametric bilinear surface for \( u, v \in [0,1] \) is:

\[
P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}
\]

Ruled Surfaces

- Given two curves \( a(t) \) and \( b(t) \) corresponding ruled surface is constructed by connecting curves with straight lines

\[
S(u,v) = va(u) + (1-v)b(u)
\]

Questions:
- When is a ruled surface a bilinear patch?
- When is a bilinear patch a ruled surface?
Boolean Sum/Coons Patch (1967)

Given four connected curves $C_i$, $i=1,2,3,4$ Boolean sum $S(u,v)$ fills the interior with surface

$$S_i(u,v) = vC_i(u) + (1-v)C_2(u)$$
$$S_2(u,v) = uC_2(v) + (1-u)C_4(v)$$

$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$

$$S(u,v) = S_i(u,v) + S_2(u,v) - P(u,v)$$

$S(u,v)$ coincides with $C_i$ along its boundaries

Examples
Tensor Product Surfaces

More General Parametric Surfaces

• Use basis functions like for curves
• Apply independently to parametric directions s and t
• Works for arbitrary basis

Example:

• Bézier curve:
\[ F(t) = \sum_{i=0}^{m} B_i^m(t) \cdot b_i \]

• Tensor product Bézier patch:
\[ F(s, t) = \sum_{i=0}^{m_i} \sum_{j=0}^{m_j} B_i^{m_i}(s) \cdot B_j^{m_j}(t) \cdot b_{i,j} \]

Clicker question

What kind of surface best describes the shape on the right?

A. Extrusion
B. Revolution
C. Sweep
D. Coons Patch
E. Ruled Surface
Clicker question

What kind of surface best describes the shape on the right?

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