CPSC 424
Bezier Triangles

Syllabus

Curves in 2D and 3D
Properties of Curves and Surfaces
Surfaces
• Sweeping/extrusion, surfaces of revolution
• Tensor-product surfaces
• Bézier triangles
• Polygonal meshes, mesh data structures
• Subdivision
Bézier Triangles

*Idea:*
- Use similar idea that lead to definition of Bézier curves to define surface with
  - *Parameterization over triangle*
  - *Small total degree*

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**Parameterization**
- Searching for
  - *Coordinate system over triangle*
  - *Should be symmetric in all vertices*
Bézier Triangles

**Barycentric Coordinates:**

\[ p = \alpha v_0 + \beta v_1 + \gamma v_2; \alpha + \beta + \gamma = 1 \]

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Polynomial Basis Over Triangles

**Bernstein Polynomials**

- **2D:**
  \[ B^m_i(t) := \binom{m}{i} t^i (1-t)^{m-i}; i = 0..m; t \in [0,1] \]

- **3D:**
  \[ B^m_{ijk}(\alpha, \beta, \gamma) := \binom{m}{i, j, k} \alpha^i \beta^j \gamma^k; i + j + k = m; i, j, k \geq 0 \]

where

\[ \binom{m}{i, j, k} := \frac{m!}{i! j! k!} \]
Bezier Triangle Control Polyhedron

De Casteljau for Bézier Triangles

De Casteljau:
De Casteljau for Bézier Triangles

**Subdivision**

- Evaluation at Barycentric coordinates (1/3, 1/3, 1/3) gives subdivision of original triangle into 3 new ones
- Problem: edges are never subdivided, triangle shape becomes worse and worse
- Solution: use multiple de Casteljau steps to split triangle into 4 patches

**Continuity**

**Approach:**
- Use derivatives to obtain constraints along edges of control mesh
- Need degree 5 Bézier triangle for $C^1$ continuity!
  - Over-constrained for lower degrees
Continuity

- For C2 many different schemes available
  - *Different advantages and disadvantages (low degree vs. patch shape vs. continuity)*
- Continuity around corners is even more complex
  - *Very hard to get continuous surfaces over triangle meshes*

Surfaces (so far)

**Tensor Product Surfaces:**
- Simple
- Always over rectangular domain
- Singularities when trying co-locating control points to simulate other domain shapes

**Bézier Triangles:**
- Simple
- Continuity is a problem

**Not discussed:**
- Triangular splines: really complex & hard to control, not very successful
Surfaces – differential geometry

Tangent plane to surface $S(u,v)$ is spanned by two partials of $S$:

$$
\frac{\partial S(u,v)}{\partial u}, \frac{\partial S(u,v)}{\partial v}
$$

Normal to surface

\[ \vec{n} = \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v} \]

• perpendicular to tangent plane

Any vector in tangent plane is tangential to $S(u,v)$

Curvature

Normal curvature of surface is defined for each tangential direction

Principal curvatures $K_{min}$ & $K_{max}$: maximum and minimum of normal curvature

• Correspond to two orthogonal tangent directions
  – Principal directions
• Not necessarily partial derivative directions
• Independent of parameterization
3D Curvature

**Isotropic**
Equal in all directions
- spherical
- planar

**Anisotropic**
2 distinct principal directions
- elliptic
- parabolic
- hyperbolic

**Principal Directions**
- min curvature
- max curvature
Curvature

Typical measures:

- **Gaussian** curvature
  
  \[ K = k_{\text{min}} k_{\text{max}} \]

- **Mean** curvature
  
  \[ H = \frac{k_{\text{min}} + k_{\text{max}}}{2} \]