CPSC 424
From Curves to Surfaces

Syllabus

Curves in 2D and 3D

Properties of Curves and Surfaces

Surfaces

- Sweeping, extrusion, surfaces of revolution
- Tensor-product surfaces
- Bézier triangles

Surfaces

Categories:

- Explicit $z = F(x, y)$
- Implicit $F(x, y, z) = 0$
- Parametric $F(s,t): \mathcal{R}^2 \mapsto \mathcal{R}^3$

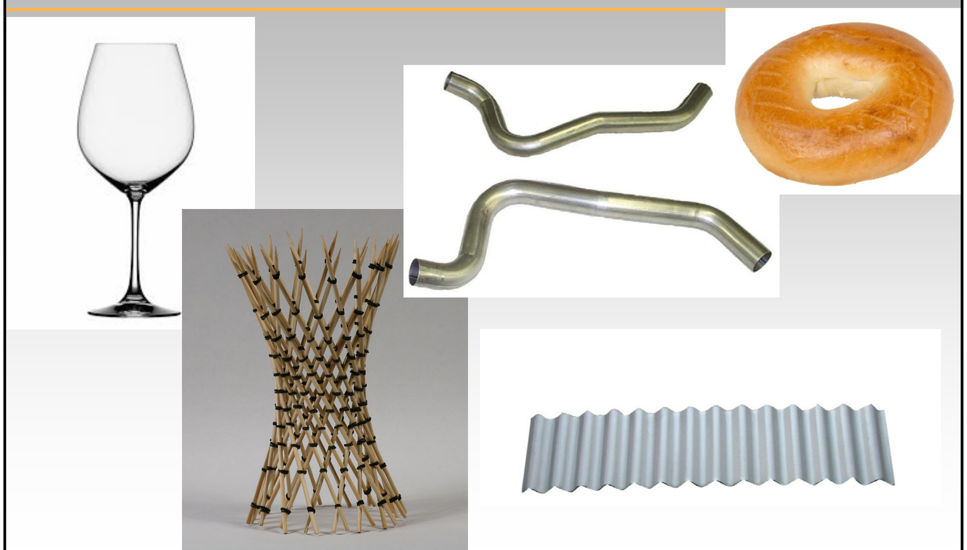
$$F(s,t) := \begin{pmatrix} F_x(s,t) \\ F_y(s,t) \\ F_z(s,t) \end{pmatrix}$$

From Curves to Surfaces

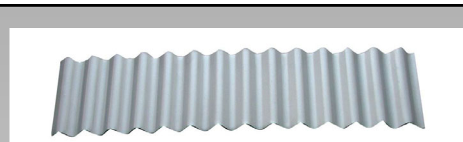
Motivation

- Derive surface shapes from curves through set of simple operations
  - *Extrusion*
  - *Revolution*
  - *Sweep*
  - *Ruling*
- The curves can be modeled using any of the techniques we have discussed

Basic surfaces

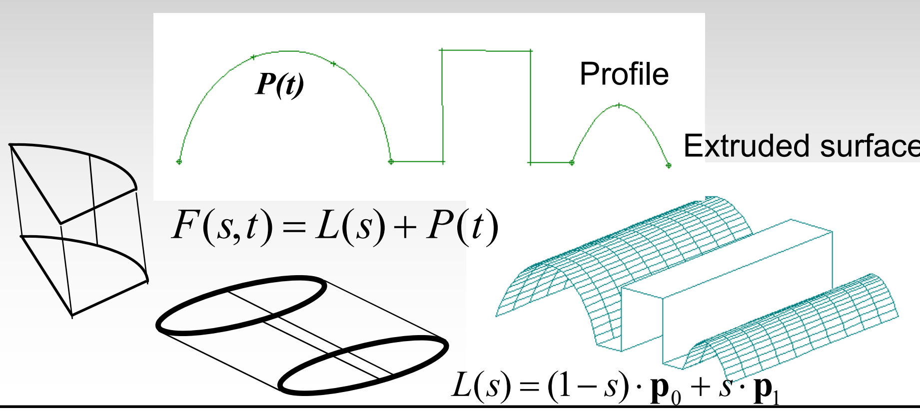


Extrusion



Concept:

- Move a curve ("profile") along a line segment
- The union of all points visited defines the surface



Extrusion as Parametric Surface

Curves:

- Profile: arbitrary curve *P(t)*
- Line segment for sweeping:

$$L(s) = (1-s)\cdot \mathbf{p}_0 + s\cdot \mathbf{p}_1$$

Swept Surface:

$$F(s,t) = L(s) + P(t)$$

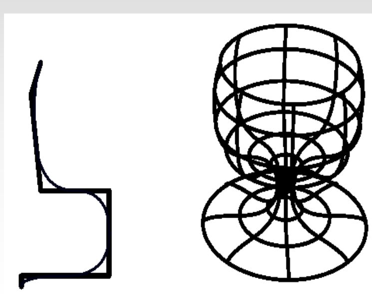
- Parametric function has very specific structure
- Generalized sweeping along arbitrary curve more complex

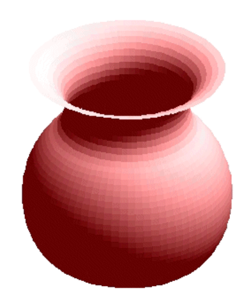
Surfaces of Revolution

Concept:

- Rotate profile curve around an axis
- $R(v)$ rotation matrix (v in $[0,2\pi]$)

$$S(u,v) = R(v)F(u)$$

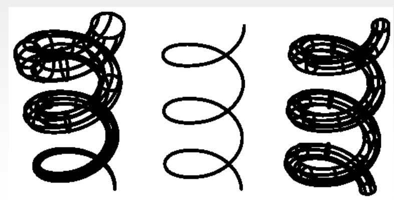




Sweeping

Concept:

- Generalize extrusion & revolution - sweep along arbitrary curve
- To orient profile at any point
  - *user specified*
  - *use Frenet frame*



Bilinear Patches

Bilinear interpolation of 4 3D points - 2D analog of 1D linear interpolation between 2 points in the plane

Given $P_{00}, P_{01}, P_{10}, P_{11}$ - associated parametric bilinear surface for $u, v \in [0,1]$ is:

Bilinear Patches

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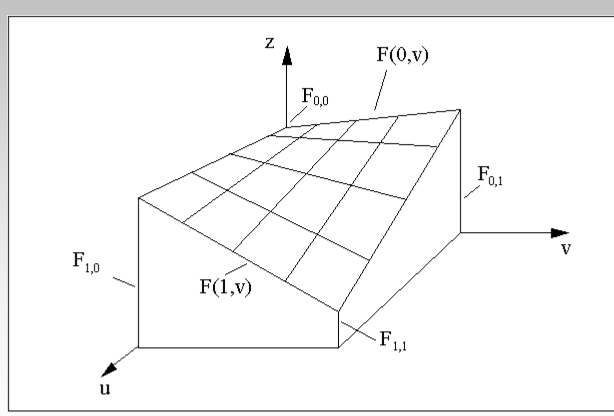
Given $P_{00}, P_{01}, P_{10}, P_{11}$ - associated parametric bilinear surface for $u, v \in [0,1]$ is:

$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$

Bilinear Patch



Bilinear Patches

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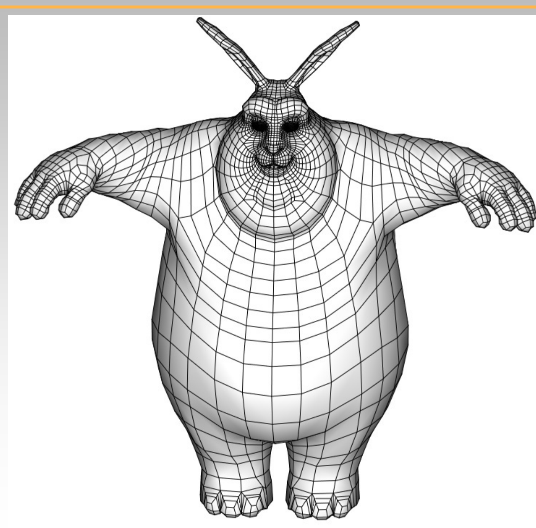
$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$

Questions:

- What does an isoparametric curve of a bilinear patch look like ?
- When is a bilinear patch planar?

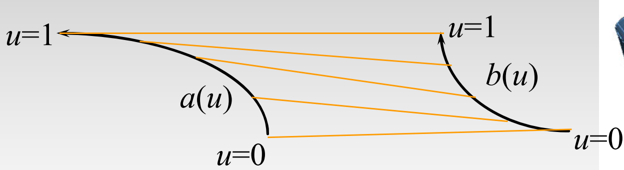
Quad Mesh: Union of Bilinear Patches

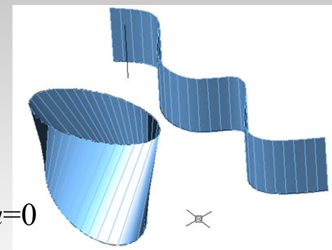


Ruled Surfaces

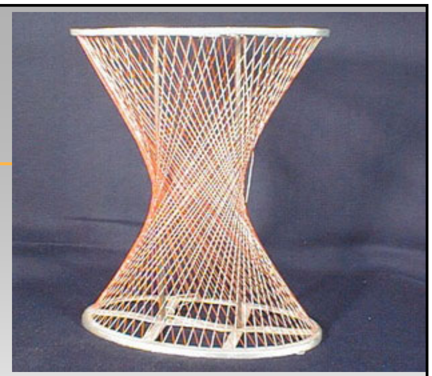
- Given two curves a(t) and b(t) corresponding ruled surface is constructed by connecting curves with straight lines

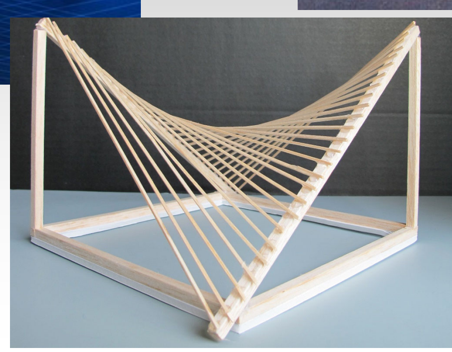
$$S(u,v)=va(u)+(1-v)b(u)$$





Ruled Surfaces





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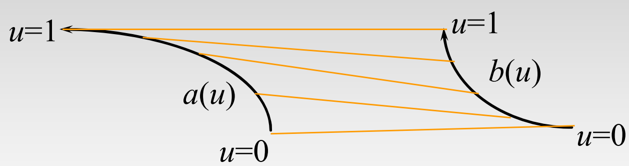
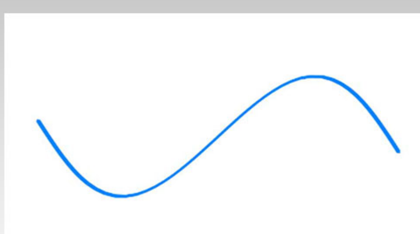


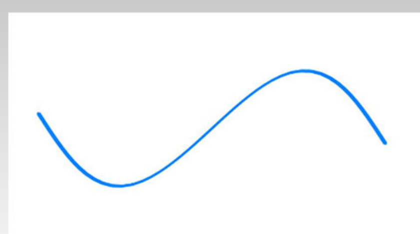
Questions:

- When is a ruled surface a bilinear patch ?
- When is a bilinear patch a ruled surface ?

Boolean Sum/Coons Patch (1967)

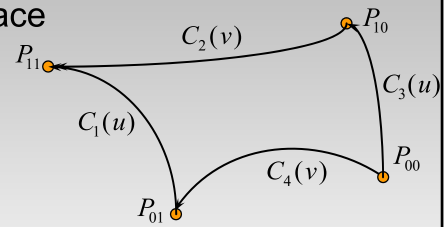
Interpolate four-sided curve loop





Boolean Sum/Coons Patch (1967)

Given four connected curves Ci I=1,2,3,4 Boolean sum S(u,v) fills the interior with surface

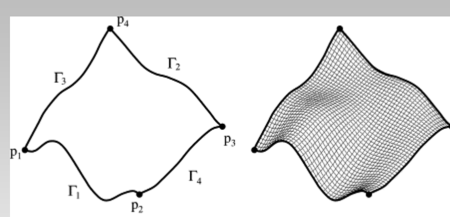


$$S_1(u,v) = vC_1(u) + (1-v)C_3(u)$$
$$S_2(u,v) = uC_2(v) + (1-u)C_4(v)$$
$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$

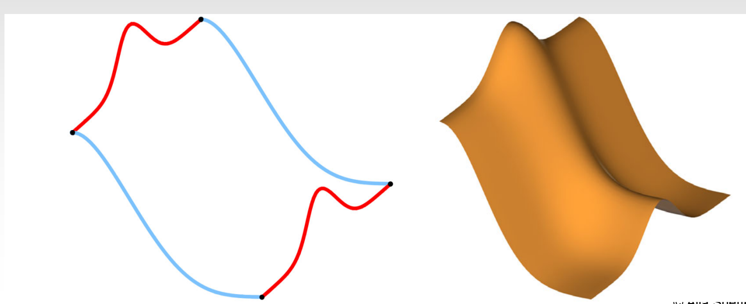
$$S(u,v) = S_1(u,v) + S_2(u,v) - P(u,v)$$

S(u,v) coincides with Ci along its boundaries

Examples



(a) (b)



Summary

- Simple methods for generating surfaces from curves
- Curves can be modeled any way we want
- Limited set of shapes

Tensor Product Surfaces

More General Parametric Surfaces

- Use basis functions like for curves
- Apply independently to parametric directions s and t
- Works for arbitrary basis

Example:

- Bézier curve:

$$F(t) = \sum_{i=0}^{m} B_i^m(t) \cdot \mathbf{b}_i$$

- Tensor product Bézier patch:

$$F(s,t) = \sum_{i=0}^{m_s} \sum_{j=0}^{m_t} B_i^{m_s}(s) \cdot B_j^{m_t}(t) \cdot \mathbf{b}_{i,j}$$

Tensor Product Surfaces

Notes:

- Surface is (rational) polynomial in s and t, depending on basis
  - *The degree in s is m_s*
  - *The degree in t is m_t*
  - *The total degree is $m_s + m_t$*
- Algorithms from curves transfer directly to tensor product surfaces
- Properties of surfaces directly related to properties of corresponding curves

Tensor Product Surfaces

Properties (Bézier, B-Spline, Rational Bézier/B-Spline):

- (Local) Convex hull
- Affine invariance
- Control points of the edge curves are the boundary points of the control mesh
- Bézier patch interpolates corner vertices of its control mesh

Tensor Product Surfaces

Continuity

- Two patches

$$F(s,t):[s_0,s_1]\times[t_0,t_1],$$
$$G(s,t):[s_1,s_2]\times[t_0,t_1]$$

- are C^k continuous if for all t

$$F^{(l)}(s,t)=G^{(l)}(s,t); l\le k$$

- Same for s
- Special case – two patches sharing one corner

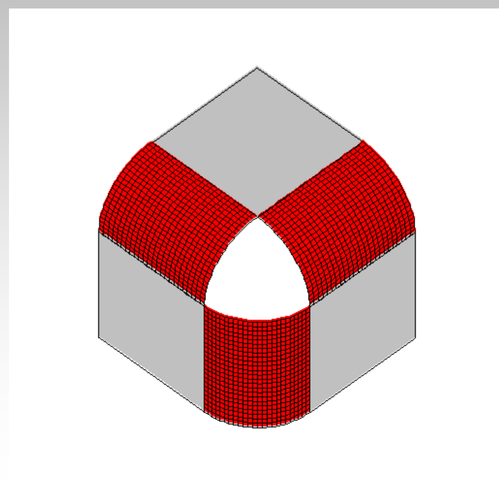
Tensor Product Surfaces

Limitations:

- Total degree is sum of degrees in s and t
- Always parameterized over rectangular parameter interval
- Refinement (degree elevation or knot insertion) always affects a whole row or column

Tensor Product Surfaces

Limitations: “suitcase corners”



Patch Networks