CPSC 424
Differential Geometry of Curves

Syllabus

Curves in 2D and 3D

- …
- Subdivision Curves

Properties of Curves and Surfaces

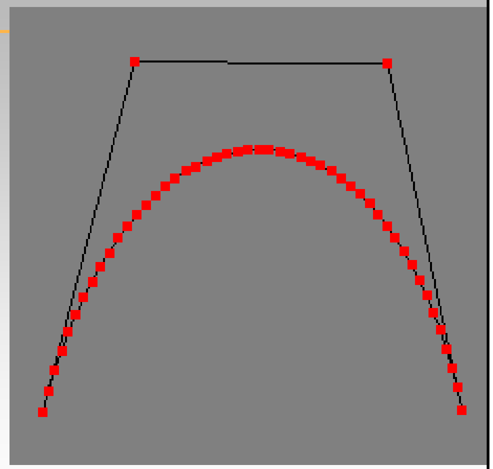
- **Differential Geometry**:
  - *arc length*
  - *curvature*
  - *Frenet frame*

Surfaces

Motivation

Parameterization:

- Curve representations we have looked at have fixed parameterization
  - *Bézier*
  - *B-Spline*
  - *Rational*
- Indicates fixed speed for point traveling along the curve



Want:

- Change speed without affecting shape

Motivation

Continuity between curves

- So far included **parameterizations**
  - *Speed wrt to input parameter*
- Example:
  - *Bézier continuity constraints depend on relative interval lengths*

Want:

- Geometric concept for continuity that ignores parameterization

Motivation

Differential Geometry helps to:

- Understand parameterizations
- Understand impact of parameterizations on derivatives
  - *Tangent vector, curvature, …*
- Change parameterization as required
- Derive geometric definition of continuity

For our purposes:

- Look at 3D curves
- 2D as special case

Regularity

Definition:

- Differentiable parametric curve $F(t)\colon [\mathrm{a},\mathrm{b}] \to \mathbb{R}^3$ is called regular if

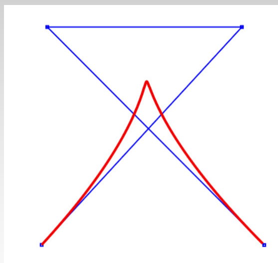
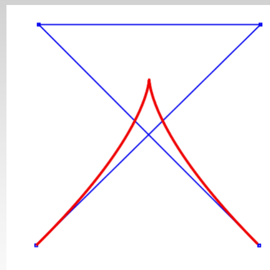
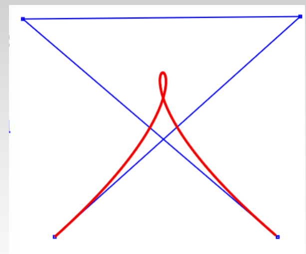
$$F'(t) \neq 0, \forall t \in [a,b]$$

- (I.e. if the tangent vector is not 0 anywhere)

Note:

- Bézier curves not necessarily regular…

Not Regular Bezier Curve

Equivalence/Reparameterization

Definition

- Two regular curves

$$F(t): [a, b] \rightarrow \mathbb{R}^3 \text{ and } G(t): [c, d] \rightarrow \mathbb{R}^3$$

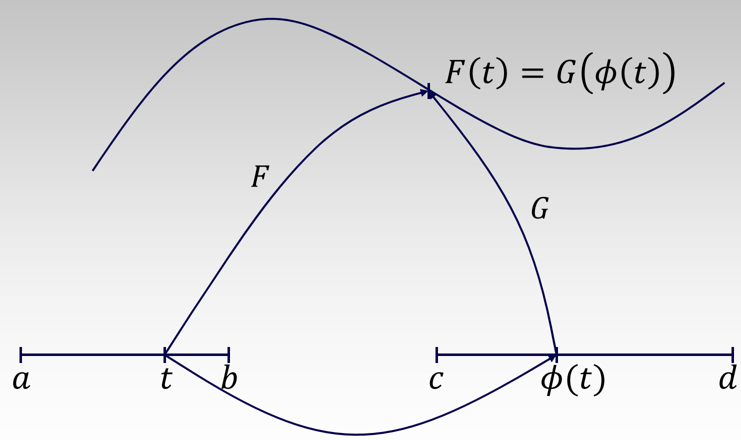
are geometrically equivalent $F \cong G$ if there is a strictly monotonic, differentiable function

$$\phi(t): [a, b] \rightarrow [c, d]$$

with

$$F(t) = G\big(\phi(t)\big)$$

Equivalence/Reparameterization



Equivalence/Reparameterization

Are the curves

$$F(t) = (2\cos t, 2\sin t) \;\; t \in [0, \pi]$$

and

$$G(s) = \left(s, \sqrt{4 - s^2}\right) \;\; s \in [-2, 2]$$

geometrically equivalent?

$$\phi(t) = ?$$

Arc Length

Definition

- Arc length of regular curve $F(t): [a, b] \to \mathbb{R}^3$ given as

$$s(t) := \int_a^t \|F'(\tilde{t})\| \, d\tilde{t}$$

Parameterization by arc length

$$G(s) \text{ with } G\big(s(t)\big) = F(t)$$

- Note: this is a **canonical** representation for any curve
- Point is traveling along G with constant speed 1

Arc Length Example (on board)

$F(t) = (t, t) \;\; t \in [0,1]$

$F'(t) = ?$ $\qquad\qquad$ $s(t) := \int_a^t \|F'(\tilde{t})\| \, d\tilde{t}$

$F(t) = (\sin(t), \cos(t)) \;\; t \in [0, \pi]$

$F'(t) = ?$

$s(t) = ? \quad G(s) = ?$

Do at home

$$F(t) = \left(1 + \frac{3}{2}t, (t-1)^{3/2}\right) \quad t \in [1,2]$$

$F'(t) = ?$ $s(t) = ?$ $G(s) = ?$

Curvature

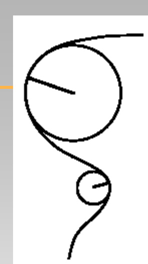
Definition

- Let G be a curve parameterized by arc length
- We introduce the following terms:
  - *Unit tangent* $T(s) \coloneqq G'(s)$
  - *Curvature vector* $K(s) \coloneqq G''(s)$
  - *Curvature* $\kappa(s) \coloneqq \|K(s)\|$
  - *Principal normal* $N(s) \coloneqq \frac{K(s)}{\kappa(s)}$
    - Up to orientation…(may need to flip for consistent frame)
  - *Binormal* $B(s) \coloneqq T(s) \times N(s)$

General Setting

- To translate into general parameterization use derivation rules
  - *e.g. Unit tangent* $T(t) := \frac{dG}{ds} = \frac{dF}{dt}\frac{dt}{ds} = \frac{F'}{\|F'\|}$

Curvature

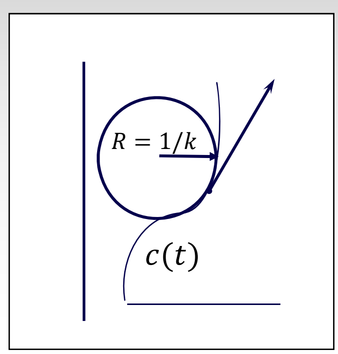


- General parameterization (2D)

$$k(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}$$

Corresponds to radius of osculating circle $R = 1/k$

Measure curve bending



Frenet Frame

Theorem:

- Curvature vector and tangent vector are perpendicular:

$$K(s) \perp T(s)$$

Note:

- Therefore, T, N, and B form an orthonormal coordinate frame
- This is called the Frenet Frame

Torsion

$$B'(s) = \tau(s)B(s)$$

Note:

- B' is the torsion vector
- τ is the torsion, and indicates how much the curve twists out of the plane ($\tau = 0$ means perfectly planar)

Fundamental Theorem of Curves

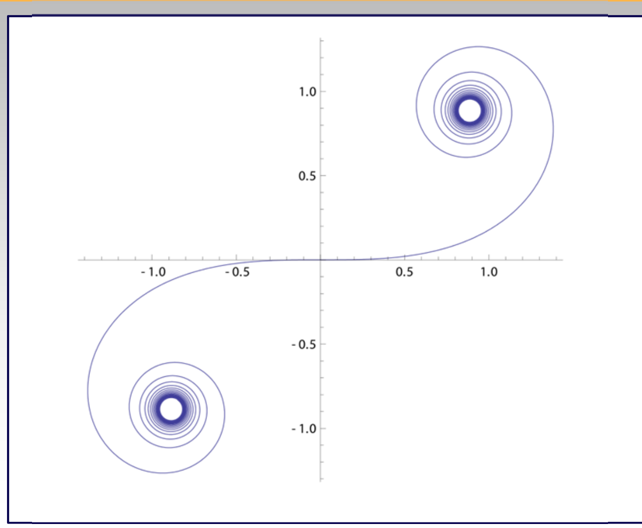
Theorem:

- For given functions $\kappa(s)$, $\tau(s)$ there exists exactly one (except for rotations and translations) unique curve that is parameterized by arc length and has curvature $\kappa(s)$, and torsion $\tau(s)$

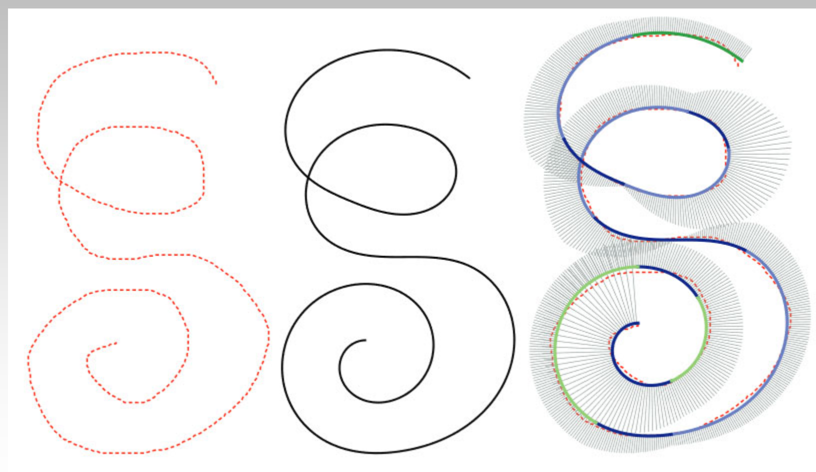
Proof:

- Quite complex, see for example
  - *Da Carmo*
    Differential Geometry of Curves and Surfaces

Clothoids



Clothoids: Sketching



Bonus Question [1%]

Plot (use software of your choice) a 2D curve whose curvature changes **linearly** from -10 to 10

5 first correct answers will get the bonus

Geometric Continuity

Definition:

- Two curves

$$F_1(t): [a, b] \to \mathbb{R}^3 \text{ and } F_2(t): [b, c] \to \mathbb{R}^3$$

are G^k-continuous (geometrically continuous of degree k) if there are reparameterizations

$$G_1(t) \cong F_1(t) \text{ and } G_2(t) \cong F_2(t)$$

that are C^k continuous, i.e.:

$$G_1^l(t) = G_2^l(t), \qquad l = 0 \dots k$$

at shared parameter interval endpoint

Geometric Continuity

Note:

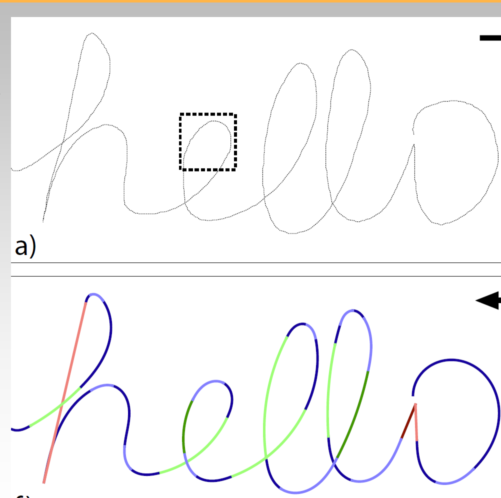
- In practice, if two curves are G^k continuous, then there always is a reparameterization for F_2 that is C^k continuous to F_1
  - *I.e. only one of the two curves needs to be reparameterized*
- If we reparameterize both curves, we can normalize the tangent vector at the transition point to unit length
  - *Local arc length parameterization*

Real-life problems - Curves

- **Reconstruction:** given (many) points sampled from smooth (or not) curve find B-spline/NURBS/Bezier/other curve that interpolates/approximates input points
  - *use A LOT less control then input points*
  - *optimize "fairness" (e.g. integral squared curvature)*
  - *support noise/outliers*
  - *Minimize number of spline segments*

Real-life problems - Curves

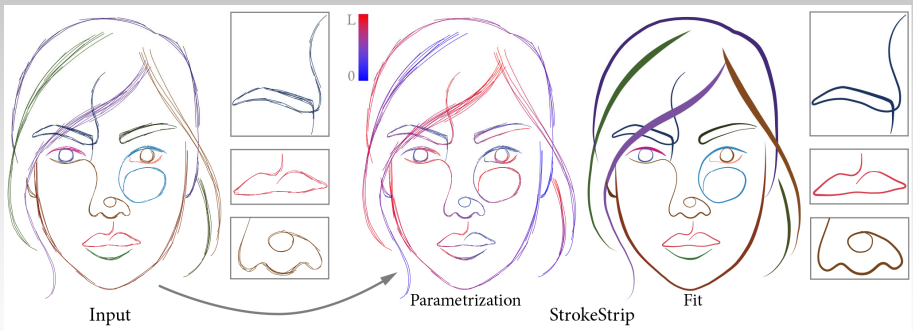
Fairing: given manually drawn (noisy) curve recover artist intended clean curve



[Baran'2010]

Real-life problems - curves

- Fitting to stroke clusters/consolidation



[Pagurek’20]

Real-life problems - Curves

- **Editing:** given B-spline/NURBS/Bezier/other curve provide simple way for user to deform it
  - *allow to pick random point on curve and pull it so the rest follows in intuitive manner*
- **Modeling:** intuitive way to form desired curves
  - *more “user-friendly” then control-point editing*

Real-life problems - Curves

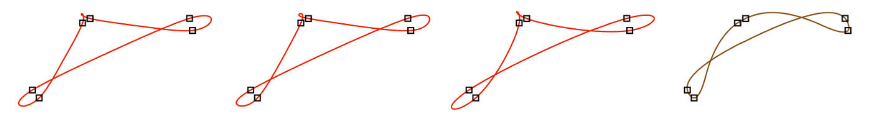


Fig. 2. Comparison of our result with other C^2 curves. From left to right: 6-point interpolatory subdivision curve [Deslauriers and Dubuc 1989], C^2 Catmull-Rom spline [Catmull and Rom 1974], C^2 interpolating cubic B-spline [Farin 2002], our curve.

[Yan'17]