CPSC 424
B-Splines

Syllabus

Curves in 2D and 3D

- Implicit vs. Explicit vs. Parametric curves
- Bézier curves
- Continuity,
- **B-Splines**
- Subdivision Curves

Properties of Curves and Surfaces

Surfaces

B-Splines

Problem with Bézier continuity:

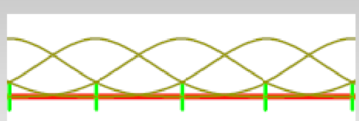
- Use more control points than degrees of freedom if we want continuity
  - *Require users to think about "correct" control placement*

Idea:

- Use different Basis (piece-wise polynomial)

B-Splines

Idea: Generate basis where functions are continuous cross domains



Control point controls set of basis functions (to preserve continuity)

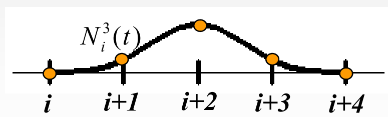
Alternative view: continuous basis functions defined on several domains

Uniform Cubic B-Spline Curves

Definition

$$C(t)=\sum_{i=0}^{n-1} P_i N_i^3(t) \quad t\in[3,n]$$

$$N_i^3(t)=\begin{cases} r^3/6 & r=t-i \quad t\in[i,i+1] \\ (-3r^3+3r^2+3r+1)/6 & r=t-i-1 \quad t\in[i+1,i+2] \\ (3r^3-6r^2+4)/6 & r=t-i-2 \quad t\in[i+2,i+3] \\ (1-r)^3/6 & r=t-i-3 \quad t\in[i+3,i+4] \end{cases}$$



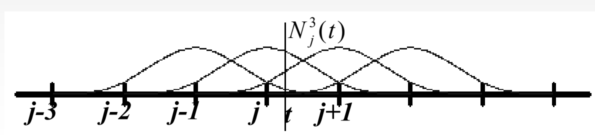
Uniform Cubic B-Spline Curves

For any t ∈ [3,n] $\sum_{i=j-3}^{j} N_i^3(t)=1$

For any t ∈ [j,j+1] only 4 basis functions are non zero

$$\sum_{i=0}^{n-1} N_i^3(t)=1$$

Any point on cubic B-Spline is affine combination of at most 4 control points



Boundary Conditions for B-Splines

B-Splines do not interpolate any control points

- in particular end points

Way to force endpoint interpolation:

- Let $P_0 = P_1 = P_2$ and same for other end

Question:

- What is the shape of the curve at endpoints if this method is used ?

B-Splines

Direct recursion formula:

$$N_i^0(t) = \begin{cases} 1 & ; u_i \le t < u_{i+1} \\ 0 & ; \text{else} \end{cases}$$

$$N_i^l(t) = \frac{t - u_i}{u_{i+l} - u_i} \cdot N_i^{l-1}(t) + \frac{u_{i+l+1} - t}{u_{i+l+1} - u_{i+1}} \cdot N_{i+1}^{l-1}(t)$$

Note:

- Not an affine combination

Normalized B-Splines

Theorem ("partition of 1"):

- The B-Spline basis functions sum to 1

Why is this so important?

- When we define curves using this basis:

$$F(t) := \sum_i N_i^m(t) \cdot \mathbf{d}_i$$

then the weights of the control points sum up to 1

- Therefore: the curve is an affine combination of the control points
- This means affine invariance!

NURBs

B-Spline

$$C(t) = \sum_{i=0}^{n-1} P_i N_i^3(t) \quad t \in [3, n]$$

$$N_i^3(t) = \begin{cases} r^3/6 & r = t - i & t \in [i, i+1] \\ (-3r^3 + 3r^2 + 3r + 1)/6 & r = t - i - 1 & t \in [i+1, i+2] \\ (3r^3 - 6r^2 + 4)/6 & r = t - i - 2 & t \in [i+2, i+3] \\ (1-r)^3/6 & r = t - i - 3 & t \in [i+3, i+4] \end{cases}$$

Non-Uniform – different interval lengths (knots)

Rational – rational basis functions

$$C(t) = \frac{\sum_{i=0}^{n-1} w_i P_i N_i^3(t)}{\sum_{i=0}^{n-1} w_i N_i^3(t)} \quad t \in [3, n]$$