CPSC 424
Degree Elevation and Continuity

Midterm date: Feb 7
Material: Curves
Syllabus

Curves in 2D and 3D
- Implicit vs. Explicit vs. Parametric curves
- Bézier curves, de Casteljau algorithm, Polar forms
- Bézier subdivision, degree elevation
- Continuity, B-Splines
- Subdivision Curves

Properties of Curves and Surfaces

Surfaces

De Casteljau Algorithm, Again

Evaluation scheme (cubic case, $t=1/2$):

\[
\begin{array}{c}
  f(1/2,1/2,1/2) \\
  f(0,1/2,1/2) \quad f(1,1/2,1/2) \\
  f(0,0,1/2) \quad f(0,1,1/2) \quad f(1,1,1/2) \\
  f(0,0,0) \quad f(0,0,1) \quad f(0,1,1) \quad f(1,1,1)
\end{array}
\]
Subdivision Example

Cubic case:

Subdivision Algorithm

Algorithm:
• Recursively subdivide control polygon at center of parameter interval
• Resulting control polygons converge to actual curve

Theorem:
• Convergence is very fast
  – for $n$ subdivision steps, the error (maximum distance between control polygons and curve) is
  \[ \varepsilon < \frac{c}{2^n} \]
  for some constant $c$
Derivatives of Bézier Curves

**Theorem (proof on board):**

- The derivative of a Bézier curve

\[
F(t) := \sum_{i=0}^{m} B_i^m(t) \cdot b_i
\]

is given as

\[
F'(t) := m \cdot \sum_{i=0}^{m-1} B_i^{m-1}(t) \cdot (b_{i+1} - b_i)
\]

Derivatives on Bézier Curves

**Note:**

- In particular, derivative can itself be interpreted as Bézier curve

- Control “points” of this “curve” are really vectors (directions)

- With this theorem we have finally shown that a Bézier curve is tangential in the control polygon at its first and last vertex.
Splines

Concept:

• Provide local control by piecing together multiple (polynomial) curves in smooth fashion

• This is called Spline

• Like with polynomial curves there are multiple representations, e.g.:
  – Hermite Spline (week 2)
  – B-Spline (next lecture)

Splines

Problem:

• How do we ensure smooth (continuous) transition between segments?

Several approaches:

• Use Bézier curves, but restrict location of some control points
  – User has to do “The Right Thing”™

• Use representation in which continuity is implicitly guaranteed
  – B-Splines
Continuity

**Tangent Vector of Parametric Curve:**

- Given by its derivative (here: 3D):
  \[ T(t) = F'(t) = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} \]

- Example (2D explicit curve):
  \[ F(t) := \begin{pmatrix} t \\ t^2 \end{pmatrix} \quad T(t) := F'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \]

**Note:**

- This is a tangent vector, not a tangent line
- This vector has a length
  - Represents speed of an object moving along the curve
Continuity

**Def:**
- A curve $F(t)$ is called $C^k$-continuous if its $k^{th}$ derivative $F^{(k)}(t)$ exists (i.e. is continuous) everywhere

**Note:**
- Polynomial curves are infinitely continuous

**Def:**
- Two curve segments $F(t)$ and $G(t)$ are called $C^k$-continuous at $t_0$ if their first $k$ derivatives match at $t_0$

**Examples:**
- $C^0$-continuous: $F(t_0) = G(t_0)$
  - i.e. the curves meet in one point for the same parameter value
- $C^1$-continuous: $F(t_0) = G(t_0)$ and $F'(t_0) = G'(t_0)$
  - They meet and have the same tangent vector
  - Note: both direction and length of vector are important!
- $C^2$-continuous: in addition $F''(t_0) = G''(t_0)$
  - Curvatures match as well
- …
C1 and C2 continuity
C1 and C2 Continuity

- **C0**: share end control points $b_m = b_0'$
- **C1**: $b_m - b_{m-1} = b_1' - b_0'$
- **G1**: $b_m - b_{m-1}$ collinear to $b_1' - b_0'$

Bezler Continuity

- **C0**: share end control points $b_m = b_0$
- **C1**: $b_m - b_{m-1} = b_1' - b_0'$
- **G1**: $b_m - b_{m-1}$ collinear to $b_1' - b_0'$