CPSC 424
Degree Elevation and Continuity

Midterm date: Feb 7

Material: Curves
Syllabus

Curves in 2D and 3D
- Implicit vs. Explicit vs. Parametric curves
- Bézier curves, de Casteljau algorithm, Polar forms
- Bézier subdivision, degree elevation
- Continuity, B-Splines
- Subdivision Curves

Properties of Curves and Surfaces

Surfaces

De Casteljau Algorithm, Again

Evaluation scheme (cubic case, t=1/2):

\[
\begin{align*}
  f(1/2,1/2,1/2) & \\
  f(0,1/2,1/2) & f(1,1/2,1/2) \\
  f(0,0,1/2) & f(0,1,1/2) & f(1,1,1/2) \\
  f(0,0,0) & f(0,0,1) & f(0,1,1) & f(1,1,1)
\end{align*}
\]
Subdivision Example

Cubic case:

Subdivision Algorithm

Algorithm:
- Recursively subdivide control polygon at center of parameter interval
- Resulting control polygons converge to actual curve

Theorem:
- Convergence is very fast
  - for n subdivision steps, the error (maximum distance between control polygons and curve) is
    \[ \varepsilon < \frac{c}{2^n} \]
  for some constant c
Derivatives of Bézier Curves

**Theorem (proof on board):**

- The derivative of a Bézier curve

\[ F(t) := \sum_{i=0}^{m} B_i^m(t) \cdot b_i \]

is given as

\[ F'(t) := m \cdot \sum_{i=0}^{m-1} B_i^{m-1}(t) \cdot (b_{i+1} - b_i) \]

Derivatives on Bézier Curves

**Note:**

- In particular, derivative can itself be interpreted as Bézier curve
- Control “points” of this “curve” are really vectors (directions)
- With this theorem we have finally shown that a Bézier curve is tangential in the control polygon at its first and last vertex.
Splines

**Concept:**
- Provide local control by piecing together multiple (polynomial) curves in smooth fashion
- This is called Spline
- Like with polynomial curves there are multiple representations, e.g.:
  - Hermite Spline (week 2)
  - B-Spline (next lecture)

Splines

**Problem:**
- How do we ensure smooth (continuous) transition between segments?

**Several approaches:**
- Use Bézier curves, but restrict location of some control points
  - *User has to do “The Right Thing”™*
  - (can be made transparent)
- Use representation in which continuity is implicitly guaranteed
  - B-Splines
Continuity

Tangent Vector of Parametric Curve:

- Given by its derivative (here: 3D):

\[ T(t) = F'(t) = \begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \end{pmatrix} \]

- Example (2D explicit curve):

\[ F(t) := \begin{pmatrix} t \\ t^2 \end{pmatrix} \quad T(t) := F''(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \]
Continuity

**Note:**
- This is a tangent vector, not a tangent line
- This vector has a length
  - Represents speed of an object moving along the curve

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Continuity

**Def:**
- A curve $F(t)$ is called $C^k$-continuous if its $k^{th}$ derivative $F^{(k)}(t)$ exists (i.e. is continuous) everywhere

**Note:**
- Polynomial curves are infinitely continuous

**Def:**
- Two curve segments $F(t)$ and $G(t)$ are called $C^k$-continuous at $t_0$ if their first $k$ derivatives match at $t_0$
Continuity

**Examples:**

- **C⁰-continuous:** \( F(t_0) = G(t_0) \)
  - *i.e. the curves meet in one point for the same parameter value*

- **C¹-continuous:** \( F(t_0) = G(t_0) \) and \( F'(t_0) = G'(t_0) \)
  - *They meet and have the same tangent vector*
  - *Note: both direction and length of vector are important!*

- **C²-continuous:** in addition \( F''(t_0) = G''(t_0) \)
  - *Curvatures match as well*

- …
C1 and C2 continuity
Bezier Continuity

- \( C^0 \): share end control points \( b_m = b'_0 \)
- \( C^1 \): \( b_m - b_{m-1} = b'_1 - b'_0 \)
- \( G^1 \): \( b_m - b_{m-1} \) collinear to \( b'_1 - b'_0 \)

Bonus [1%]

**How to construct an interpolating \( C^2 \) continuous Bezier Spline?**

- Hint: input points = end points of Bezier segments