CPSC 424
Degree Elevation and Continuity

Midterm date: October 7

Material: Curves

Clicker Question

Can a circle be described using an explicit representation?

A. Yes

B. No

Syllabus

Curves in 2D and 3D

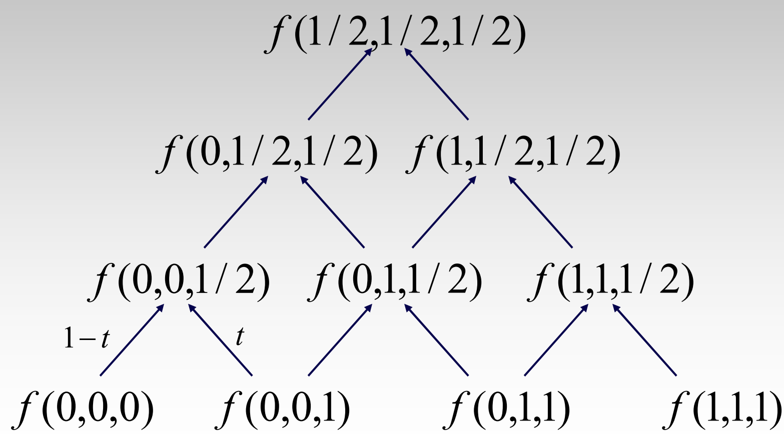
- Implicit vs. Explicit vs. Parametric curves
- Bézier curves**,** de Casteljau algorithm
- **Bézier subdivision**, **degree elevation**
- **Continuity**, B-Splines
- Subdivision Curves

Properties of Curves and Surfaces

Surfaces

De Casteljau Algorithm, Again

Evaluation scheme (cubic case, t=1/2):



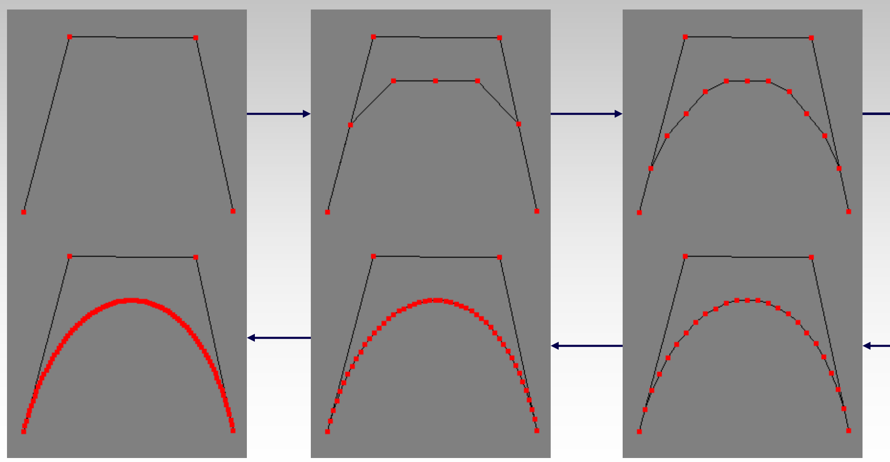
Observation

De Casteljau generates 2 new control polygons!

- For parameter interval [0,1/2], and [1/2,1]
- Can be used to recursively subdivide control polygon
- Can you prove it?

Subdivision Example

Cubic case:



Subdivision Algorithm

Algorithm:

- Recursively subdivide control polygon at center of parameter interval
- Resulting control polygons converge to actual curve

Theorem (proof in book, Section 5.2): :

- Convergence is very fast
  - *for n subdivision steps, the error (maximum distance between control polygons and curve) is*

$$\varepsilon < \frac{c}{2^n}$$

for some constant c

Derivatives of Bézier Curves

Theorem (proof on board):

- The derivative of a Bézier curve

$$F(t) := \sum_{i=0}^{m} B_i^m(t) \cdot \mathbf{b}_i$$

is given as

$$F'(t) := m \cdot \sum_{i=0}^{m-1} B_i^{m-1}(t) \cdot (\mathbf{b}_{i+1} - \mathbf{b}_i)$$

Derivatives on Bézier Curves

Note:

- In particular, derivative can itself be interpreted as Bézier curve
- Control "points" of this "curve" are really vectors (directions)
- With this theorem we have finally shown that a Bézier curve is tangential in the control polygon at is first and last vertex.

Problem: how to describe long curves?

Using high-order Bezier curve = approximation quality dimineshes

Splines

Concept:

- Provide local control by piecing together multiple (polynomial) curves in smooth fashion
- This is called Spline
- Like with polynomial curves there are multiple representations, e.g.:
  - *Hermite Spline (week 2)*
  - *B-Spline (next lecture?)*

Splines in power point

Splines

Problem:

- How do we ensure **smooth** (continuous) transition between segments?

Several approaches:

- Use Bézier curves, but restrict location of some control points
  - *User has to do "The Right Thing"*™
    - (can be made transparent)
- Use representation in which continuity is implicitly guaranteed
  - *B-Splines*

Continuity

Tangent Vector of Parametric Curve:

- Given by its derivative (here: 3D):

$$T(t) = F'(t) = \begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \end{pmatrix}$$

- Example (2D explicit curve):

$$F(t) := \begin{pmatrix} t \\ t^2 \end{pmatrix} \qquad T(t) := F'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

Continuity

Note:

- This is a tangent vector, not a tangent direction
- This vector has a length
  - *Represents speed of an object moving along the curve*

Continuity

Def:

- A curve $F(t)$ is called C^k-continuous if its k^{th} derivative $F^{(k)}(t)$ exists (i.e. is continuous) everywhere

Note:

- Polynomial curves are infinitely continuous

Def:

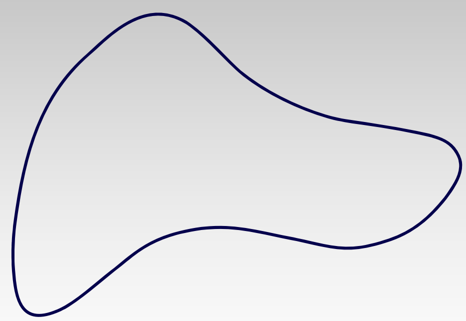
- Two curve segments $F(t)$ defined over [t,t0] and $G(t)$ defined over [t0,t'] are called C^k-continuous at t_0 if their first k derivatives match at t_0
  - *Definition extends to cases with "shifted" parameter intervals $F(t)$ and $G(t)$ are called C^k-continuous if at if first k derivatives of F(t) at t_0 match first k derivatives of G(t) at t_1*

Continuity

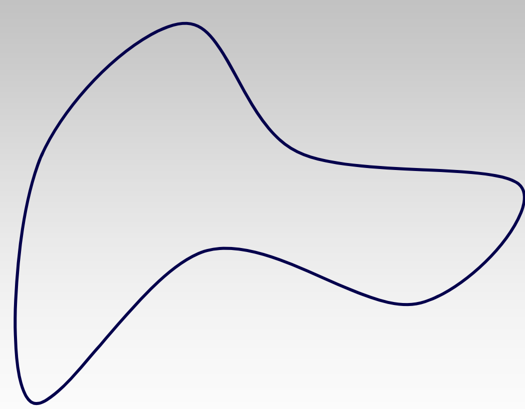
Examples:

- C^0-continuous: $F(t_0) = G(t_0)$
  - *I.e. the curves meet in one point for the same parameter value*
- C^1-continuous: $F(t_0) = G(t_0)$ and $F'(t_0) = G'(t_0)$
  - *They meet and have the same tangent vector*
  - *Note: both direction and length of vector are important!*
- C^2-continuous: in addition $F''(t_0) = G''(t_0)$
  - *Curvatures match as well*
- …

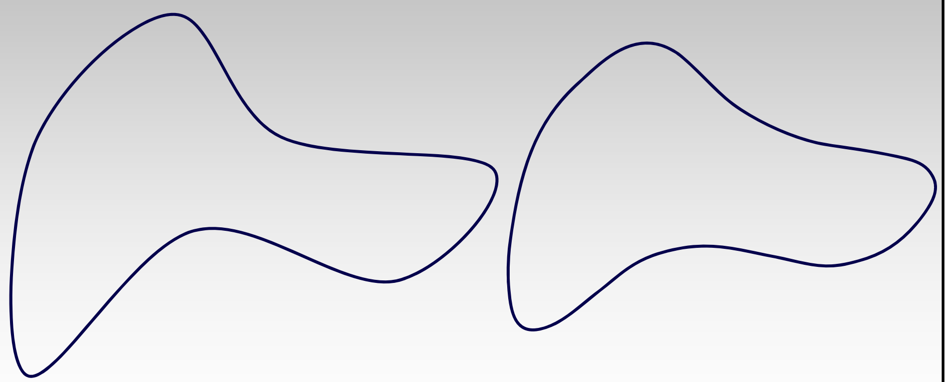
C1 and C2 continuity



C1 and C2 continuity



C1 and C2 continuity



Bezier Continuity

- ***C⁰ :*** share end control points $b_m = b'_0$
- ***C¹ :*** $b_m - b_{m-1} = b'_1 - b'_0$
- ***G¹ :*** $b_m - b_{m-1}$ collinear to $b'_1 - b'_0$

Examples (on board)

Continuity Between Bézier Curves

Remarks:

- For C^k – continuity, the first (k+1) control points of G(t) are fixed
  - *Given as affine combinations of the control points of F(t)*
  - *NOT: convex combinations*
    - There will be coefficients >1 and <0
    - Extrapolation rather than interpolation