# CPSC 424 Assignment 4

Term: September 2022, Instructor: Alla Sheffer, sheffa@cs.ubc.ca,
http://www.ugrad.cs.ubc.ca/˜cs424

## Due: November 2, 2022 in class

## Assignment 4.1: Subdivision Curves (15 Points)

Given a closed polygon defined by the sequence of points $P_i = (p_1^i, p_2^i, ..., p_n^i)$, consider the following subdivision scheme that generates after one step of subdivision the polygon $P_{i+1} = (p_1^{i+1}, p_2^{i+1}, ..., p_{2n}^{i+1})$ using the following subdivision rules:

- $p_{2j-1}^{i+1} = \frac{1}{6}p_{j-1}^i + \frac{2}{3}p_j^i + \frac{1}{6}p_{j+1}^i$
- $p_{2j}^{i+1} = \frac{1}{2}p_j^i + \frac{1}{2}p_{j+1}^i$

(Note that if $j = 1$, then $j - 1$ becomes $n$, and if $j = n$, then $j + 1$ becomes 1)

**a)** Perform 2 levels of subdivision starting from the triangle in the figure below. **Be as accurate as possible and write down the exact positions of each point.**

P1 = (0,0), P2 = (2,0), P3 = (1,1)

Level 1:

$$P_1^1 = \frac{1}{6}P_3^0 + \frac{2}{3}P_1^0 + \frac{1}{6}P_2^0 = (\frac{1}{2}, \frac{1}{6})^T$$

$$P_2^1 = \frac{1}{2}P_1^0 + \frac{1}{2}P_2^0 = (1, 0)^T$$

$$P_3^1 = \frac{1}{6}P_1^0 + \frac{2}{3}P_2^0 + \frac{1}{6}P_3^0 = (\frac{3}{2}, \frac{1}{6})^T$$

$$P_4^1 = \frac{1}{2}P_2^0 + \frac{1}{2}P_3^0 = (\frac{3}{2}, \frac{1}{2})^T$$

$$P_5^1 = \frac{1}{6}P_2^0 + \frac{2}{3}P_3^0 + \frac{1}{6}P_1^0 = (1, \frac{2}{3})^T$$

$$P_6^1 = \frac{1}{2}P_3^0 + \frac{1}{2}P_1^0 = (\frac{1}{2}, \frac{1}{2})^T$$

Level 2:

$$P_1^2 = \frac{1}{6}P_6^0 + \frac{2}{3}P_1^0 + \frac{1}{6}P_2^0 = (\frac{7}{12}, \frac{7}{36})^T$$

$$P_2^2 = \frac{1}{2}P_1^0 + \frac{1}{2}P_2^0 = (\frac{3}{4}, \frac{1}{12})^T$$

$$P_3^2 = \frac{1}{6}P_1^0 + \frac{2}{3}P_2^0 + \frac{1}{6}P_3^0 = (1, \frac{1}{18})^T$$

$$P_4^2 = \frac{1}{2}P_2^0 + \frac{1}{2}P_3^0 = (\frac{5}{4}, \frac{1}{12})^T$$

$$P_5^2 = \frac{1}{6}P_2^0 + \frac{2}{3}P_3^0 + \frac{1}{6}P_4^0 = (\frac{17}{12}, \frac{7}{36})^T$$

$$P_6^2 = \frac{1}{2}P_3^0 + \frac{1}{2}P_4^0 = (\frac{3}{2}, \frac{1}{3})^T$$

$$P_7^2 = \frac{1}{6}P_3^0 + \frac{2}{3}P_4^0 + \frac{1}{6}P_5^0 = (\frac{17}{12}, \frac{17}{36})^T$$

$$P_8^2 = \frac{1}{2}P_4^0 + \frac{1}{2}P_5^0 = (\frac{5}{4}, \frac{7}{12})^T$$

$$P_9^2 = \frac{1}{6}P_4^0 + \frac{2}{3}P_5^0 + \frac{1}{6}P_6^0 = (1, \frac{11}{18})^T$$

$$P_8^2 = \frac{1}{2}P_5^0 + \frac{1}{2}P_6^0 = (\frac{3}{4}, \frac{11}{18})^T$$

$$P_{11}^2 = \frac{1}{6}P_5^0 + \frac{2}{3}P_6^0 + \frac{1}{6}P_1^0 = (\frac{7}{12}, \frac{17}{36})^T$$

$$P_1 2^2 = \frac{1}{2}P_6^0 + \frac{1}{2}P_1^0 = (\frac{1}{2}, \frac{1}{3})^T$$

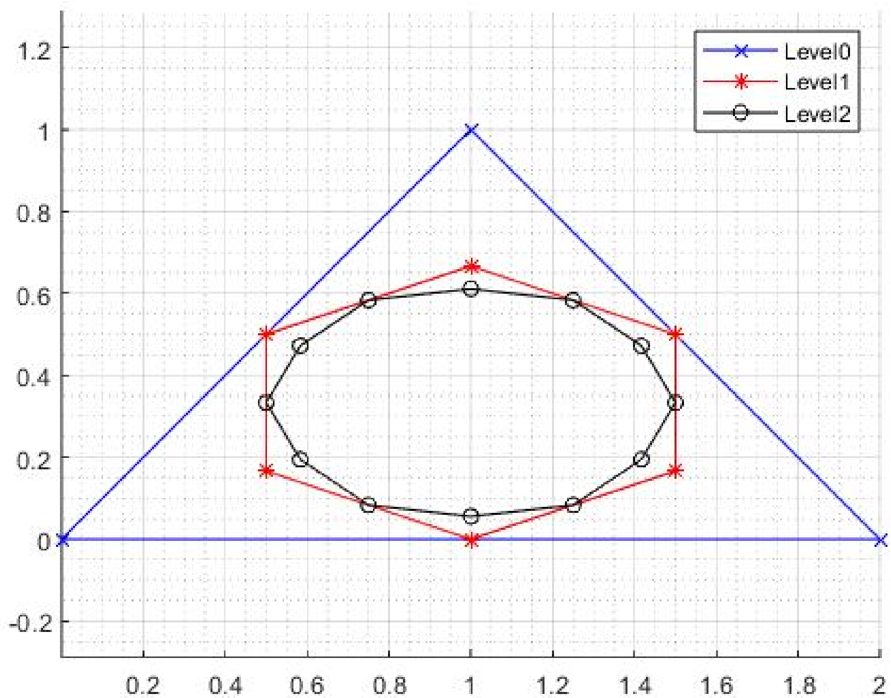



Figure 1: Two levels of subdivision

**b)** Use the formulas above to define a subdivision matrix $M$ relating $p^1_{2j-2}, p^1_{2j-1}, p^1_{2j}$ to $p^0_{j-1}, p^0_j, p^0_{j+1}$

$$\begin{bmatrix} p^1_{2j-2} \\ p^1_{2j-1} \\ p^1_{2j} \end{bmatrix} = \overbrace{\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/2 & 1/2 \end{bmatrix}}^{M} \begin{bmatrix} p^0_{j-1} \\ p^0_j \\ p^0_{j+1} \end{bmatrix}$$

**c)** Analyze the matrix $M$ (use a computer algebra system such as MATLAB if necessary) to determine whether the scheme converges.

Find the eigenvalues of the the Matrix $M$.

$$\lambda = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{6} \end{bmatrix}$$

$\lambda_1 = 1$ $|\lambda_2|$ and $|\lambda_3| < 1$
Thus, subdivision will converge.

**d)** Speculate what the positions and tangents will be at $\lim_{k\to\infty} p^k_{j\cdot 2^k}$ (limit control points corresponding to the original $p_j$). Use a computer algebra system if necessary. Sketch your results below - **be as accurate as possible**.

P1 = (0,0), P2 = (2,0), P3 = (1,1)

Position:

If we find the three eigenvalues and eigenvectors of Matrix $M$:

$$Mv_1 = \lambda_1 v_1$$

$$Mv_2 = \lambda_1 v_2$$

$$Mv_3 = \lambda_1 v_3$$

We can write them in a matrix form:

$$M \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \lambda_3 v_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Or,

$$MV = V\Lambda$$

where $\Lambda$ is the diagonal matrix of eigenvalues and V is the matrix of their corresponding eigenvectors.

Then for (here $MV = V\Lambda$),

$$A_1 = MA_0 = MVV^{-1}A_0 = V\Lambda V^{-1}A_0$$

$$A_2 = MA_1 = MV\Lambda V^{-1}A_0 = V\Lambda\Lambda V^{-1}A_0 = V\Lambda^2 V^{-1}A_0$$

...

$$A_j = M^j A_0 = V\Lambda^j V^{-1}A_0$$

$$A_\infty = \lim_{k\to\infty} M^k A_0 = V\Lambda^\infty V^{-1}A_0$$

We can write out $V$ and $\Lambda$:

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -\frac{2}{3} \\ 1 & -1 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$$

Since $\lambda_2, \lambda_3 < 1$, they will vanish while k approach infinity.

Thus,

$$\Lambda^\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then

$$M^{\infty} = V\Lambda^{\infty}V^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$$

$$A^{\infty} = M^{\infty}A_0$$

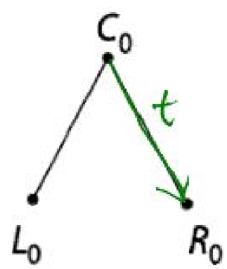
Thus,

$P_1$ will converge to $(\frac{3}{5}, \frac{1}{5})$

$P_2$ will converge to $(\frac{7}{5}, \frac{1}{5})$

$P_3$ will converge to $(1, \frac{3}{5})$

Tangent:




$$A_j = \begin{bmatrix} L_j^x & L_j^y & 1 \\ C_j^x & C_j^y & 1 \\ R_j^x & R_j^y & 1 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix}$$

$$t_j = R_j - C_j = [0 \ -1 \ 1]A_j = d^T A_j$$

$$t_j = d^T A_j = d^T V \Lambda^j V^{-1} A_0 = d^T [v_1 v_2 v_3] \Lambda^j V^{-1} A_0$$

$$t_j = [d^T v_1 \ d^T v_2 \ d^T v_3] \begin{bmatrix} \lambda_1^j & 0 & 0 \\ 0 & \lambda_2^j & 0 \\ 0 & 0 & \lambda_3^j \end{bmatrix} V^{-1} A_0$$

let,

$$[d^T v_1 \ d^T v_2 \ d^T v_3] = [w_1 \ w_2 \ w_3]$$

Then,

$$t_j = [w_1 \ w_2 \ w_3] \begin{bmatrix} \lambda_1^j & 0 & 0 \\ 0 & \lambda_2^j & 0 \\ 0 & 0 & \lambda_3^j \end{bmatrix} V^{-1} A_0$$

$$t_j = [\lambda_1^j w_1 \ \lambda_2^j w_2 \ \lambda_3^j w_3] \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix} A_0$$

$$t_j = (\lambda_1^j w_1 u_1^T + \lambda_2^j w_2 u_2^T + \lambda_3^j w_3 u_3^T) A_0$$

where,

$$w_1 = [0 \ -1 \ 1] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

Then,

$$\hat{t_j} = \frac{(\lambda_2^j w_2 u_2^T + \lambda_3^j w_3 u_3^T)}{||(\lambda_2^j w_2 u_2^T + \lambda_3^j w_3 u_3^T)||}$$

$$\hat{t_j} = \frac{(w_2 u_2^T + \frac{\lambda_3^j}{\lambda_2^j} w_3 u_3^T)}{||(w_2 u_2^T + \frac{\lambda_3^j}{\lambda_2^j} w_3 u_3^T)||}$$

$$\lim_{j \to \infty} = \frac{w_2 u_2^T A_0}{||w_2 u_2^T A_0||} = \frac{u_2^T A_0}{||u_2^T A_0||}$$

Tangent:

At $P_1$ = $< \frac{1}{2}, -\frac{1}{2} > = < \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} >$

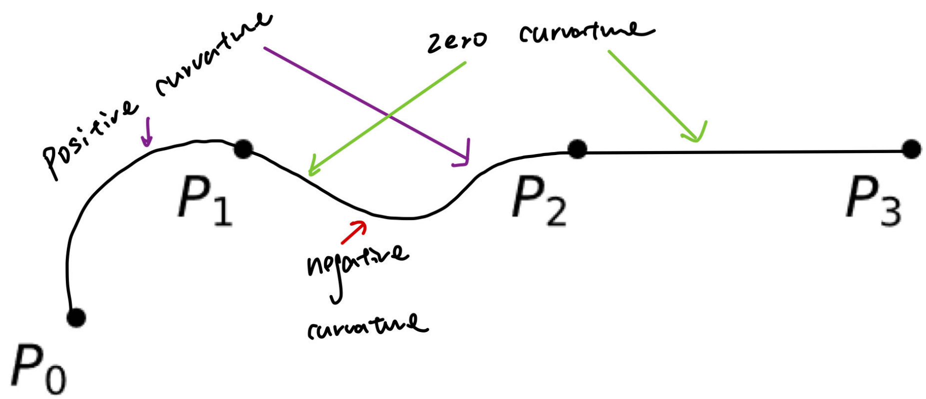
At $P_2$ = $< \frac{1}{2}, \frac{1}{2} > = < \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} >$

At $P_3$ = $< -1, 0 >$

## Assignment 4.2: Differential Geometry (10 Points)

**a)** Sketch a 2D curve with constant non-zero curvature. Explain your drawing.
A circle has constant curvature equal to $\pm 1$ divided by its radius everywhere.

**b)** Sketch a finite 2D curve which has regions of negative, positive, and zero curvature. Explain your drawing.

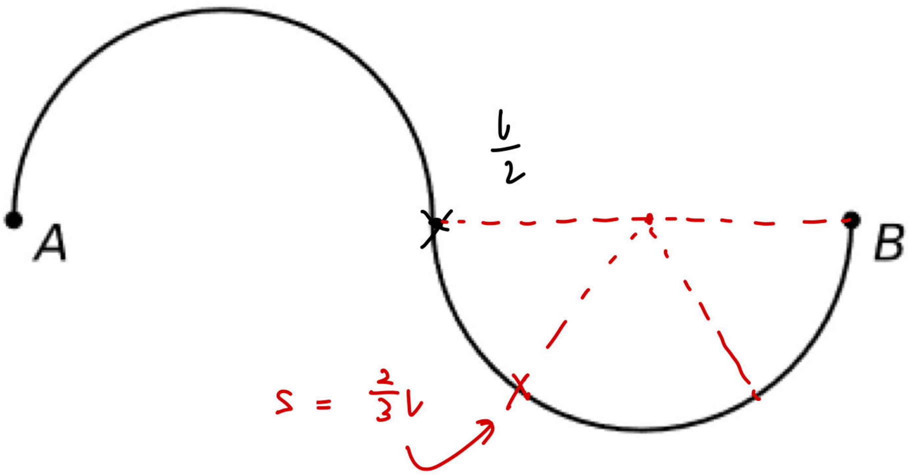



**c)** Provide a real life example of a 3D curve with non-zero torsion. Explain your choice.

Imagine a corkscrew / solenoid / slinky. There is no plane which can intersect the entire curve, thus the curve has non-zero torsion somewhere (in fact, it has non-zero torsion everywhere).

**d)** Given the curve below, which starts at point A and ends at point B, has length $l$, and is parameterized using arc-length parameterization $s \in [0, l]$ mark the point on the curve at parameter value $s = l/4$. Explain your marking.

Arc-length parameterization means that the curve is stroked at a constant speed




1. Due to symmetrical pattern, we can first find the mid point as s = $\frac{l}{2}$.
2. $\frac{2l}{3} - \frac{l}{2}\ \frac{l/6}{l/2} = \frac{1}{3}$ Then the rest takes over one third of the arc on the right hand side.

**e)** Prove that the curves $F(t) = (t, t^2 - 4)$ $t \in [0, 1]$ and $G(t) = (2t - 2, 4t^2 - 8t)$ $t \in [1, 1.5]$ are geometrically equivalent.

$$f(t) = \begin{cases} t \\ t^2 - 4 \end{cases}$$

$$g(t) = \begin{cases} 2t - 2 \\ 4t^2 - 8t \end{cases}$$

Let

$$\phi(t) = 2t - 2$$

$$\phi(t)^2 - 4 = 4t^2 - 8t$$

$$[2(t - 1)]^2 - 4 = 4(t^2 - 2t + 1) - 4 = 4t^2 - 8t$$

LHS = RHS. Thus geometric equivalence proven.

**f)** Given the curve $F(t) = (t, t)$ over the interval $t \in [0, s]$, write down the formula for the length of the curve as a function of $s$.

The curve is a straight line from $F(0) = (0, 0)$ to $F(s) = (s, s)$ and thus has length

$$\left\| \begin{bmatrix} s \\ s \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\| = \sqrt{(s)^2 + (s)^2} = \sqrt{2} \cdot s.$$