CPSC 424 Assignment 1

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Due: Sep 21/2022 in class

Assignment 1.1: Curve Representation: Lines (9 Points)

Given the points, $P_0 = (1, 0), P_1 = (5, 3), P_1 = (1, 3)$.
a) Find the *implicit* equation of an infinite straight line going through $\{P_0, P_1\}$.

Solution:
A general equation of a 2D infinite straight line is:

$$y = \frac{P_{1,y} - P_{0,y}}{P_{1,x} - P_{0,x}}\, x + c \tag{1}$$

Convert to implicit equation:

$$(P_{1,y} - P_{0,y})\, x - (P_{1,x} - P_{0,x})\, y + c = 0 \tag{2}$$

where x and y are variables. Substitute the two points into it, yielding:

$$3x - 4y + c = 0 \tag{3}$$

Then, we substitute either one of the two points to find c. After that, substitute back to the equation above which yields:

$$3x - 4y - 3 = 0 \tag{4}$$

b) Find the *implicit* equation of an infinite straight line going through $\{P_0, P_2\}$.

Solution:
A general equation of a 2D infinite straight line is:

$$y = \frac{P_{1,y} - P_{0,y}}{P_{1,x} - P_{0,x}}\, x + c \tag{5}$$

Convert to implicit equation:

$$(P_{1,y} - P_{0,y})\, x - (P_{1,x} - P_{0,x})\, y + c = 0 \tag{6}$$

where x and y are variables. Substitute the two points into it, yielding:

$$3x + c = 0 \tag{7}$$

Then, we substitute either one of the two points to find c. After that, substitute back to the equation above which yields:

$$3x - 3 = 0 \tag{8}$$

c) Convert the *implicit* equation in a) to *explicit* form, if it's possible. Show your work. If it's not possible to convert, explain your reasons.

Solution:
The explicit form of a 2D line is

$$y = a \times x + b \tag{9}$$

where x and y are also variables. By simple arrangement of the result from question (a), we have:

$$y = \frac{3}{4}x - \frac{3}{4} \tag{10}$$

d) Convert the *implicit* equation in b) to *explicit* form, if it's possible. Show your work. If it's not possible to convert, explain your reasons.

Solution:
The explicit form of a 2D line is

$$y = a \times x + b \tag{11}$$

where x and y are also variables. By simple arrangement of the result from question (b), we have:

$$x = 1 \tag{12}$$

e) Find the parametric equation for the straight line passing through $\{P_0, P_1\}$.

Solution:
The parametric equation of a 2D line takes the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \tilde{\mathbf{a}} \cdot t \tag{13}$$

where t is the parameter, $(x_0, y_0)^T$ is point on the line, and $\tilde{\mathbf{a}}$ is a vector aligned with the line. Then we can set $(x_0, y_0)^T$ to be either of the two points, e.g.,

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{14}$$

and $\tilde{\mathbf{a}}$ to be the difference of the two points, i.e.,

$$\tilde{\mathbf{a}} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{15}$$

Eventually, the parametric form is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} \cdot t \tag{16}$$

Assignment 1.2: Curve Representation: Closed Curves (12 Points)

Implicit functions are often used to separate samples identified as being inside a given shape from those identified as being outside. For instance given a set of points I in 2D space identified as inside, and a set O identified as *outside*, one can define a closed curve separating the two by devising an implicit function F that is positive for all $p \in I(F(p) > 0)$ and negative for all $p \in O(F(p) < 0)$. The separating curve is then defined by all points p where $F(p) = 0$.

a) Given the following sets $I = \{(1,1)\}, O = \{(-1,0),(0,3)\}$ devise an implicit curve F that separates them. Write down the equation of F and sketch it below.

Solution:
A circle can be used to separate the point sets. The general implicit equation of a 2D circle is:

$$x^2 + y^2 - R^2 = 0 \tag{17}$$

In order to separate the points, we could place the center of the circle at $(1,1)$ and set $R = 1$. Then, the implicit equation becomes

$$(x-1)^2 + (y-1)^2 - 1 = 0 \tag{18}$$

As stated in the question, we want all $p \in I(F(p) > 0)$ and negative for all $p \in O(F(p) < 0)$. Then, the implicit equation should be modified as:

$$-(x-1)^2 - (y-1)^2 + 1 = 0 \tag{19}$$

Tests for points:

$$F(1,1) = -(1-1)^2 - (1-1)^2 + 1 = 1 > 0 \tag{20}$$

$$F(-1,0) = -(-1-1)^2 - (0-1)^2 + 1 = -4 - 1 + 1 = -4 < 0 \tag{21}$$

$$F(0,3) = -(0-1)^2 - (3-1)^2 + 1 = -1 - 4 + 1 = -4 < 0 \tag{22}$$

The figure is as follows:

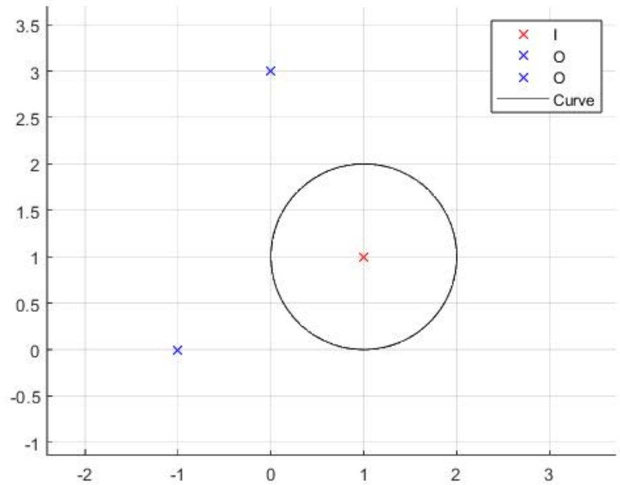

Figure 1: The point sets and implicit curve.

b) Is F unique, or are there other implicit functions that can be used to separate these sets?

Solution:
No it is not unique. There are infinite number of other curves that can separate the I and O.

c) Given the following sets $I = \{(0,0),(3,0),(6,0)\}, O = \{(-1,0),(1,0),(4,0),(7,0)\}$ devise an implicit curve F that separates them. Write down the equation of F and sketch it below.

Solution:
A $sine$ or $cosine$ curve can be used to separate the point sets. The general implicit equation of a $cosine$ curve is:

$$cos(ax - b) - y = 0 \tag{23}$$

By testing different a and b coefficients in Matlab, $a = 2.3$ $b = -0.5$ gives the result that separates the point sets. Its implicit equation is then:

$$cos(2.3x - 0.5) - y = 0 \tag{24}$$

The figure is as follows:

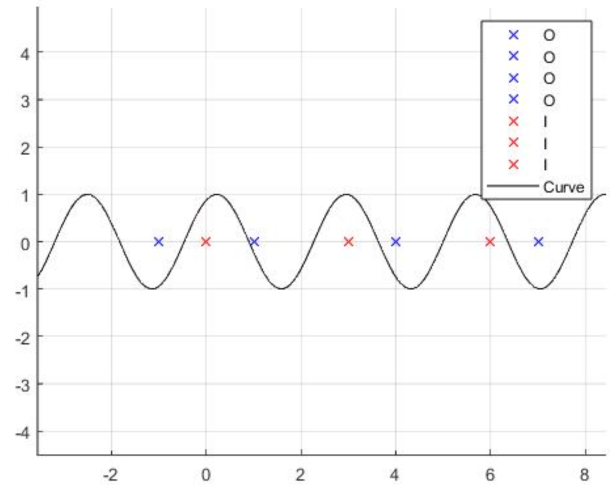

Figure 2: The point sets and implicit curve.

d) Is F unique, or are there other implicit functions that can be used to separate these sets?

Solution:
No it is not unique. There are infinite number of other curves that can separate the I and O.

Assignment 1.3: Lagrange Polynomials (10 Points)

Given a set of $k+1$ pairs of point positions $x_i \in R^N$ and corresponding parameter values t_i: $(t_0, x_0), (t_1, x_1), ..., (t_k, x_k)$, we can use the following polynomial function $F(t)$ of degree up to k to create a curve in R^N that interpolates them:

$$F(t) = \sum_{i=0}^{k} L_i(t) \cdot x_i, \tag{25}$$

and

$$L_i(t) = \prod_{0 \le m \le k, m \ne i} \frac{t - t_m}{t_i - t_m}. \tag{26}$$

a) Given two points:

- $t_0 = 0, x_0 = (3, 2)$,
- $t_1 = 2, x_1 = (6, 1)$,

write down the interpolating polynomial. Show your work. Sketch the resulting curve (hint - plug in additional values of t to get the corresponding points).

Solution:
There are only two points. Thus the degree k of the Lagrange polynomial curve is 1. That being said:

$$F(t) = L_0(t) \cdot x_0 + L_1(t) \cdot x_1 \tag{27}$$

And by expanding Eq. 26, we have

$$L_0(t) = \frac{t - t_1}{t_0 - t_1} = \frac{t - 2}{0 - 2} = \frac{2 - t}{2} \tag{28}$$

$$L_1(t) = \frac{t - t_0}{t_1 - t_0} = \frac{t - 0}{2 - 0} = \frac{t}{2} \tag{29}$$

Substitute the two points into the equations above, yielding:

$$F(t) = \frac{2 - t}{2} \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \frac{t}{2} \cdot \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{6+3t}{2} \\ 2 - \frac{t}{2} \end{pmatrix} \tag{30}$$

The figure is as follows:

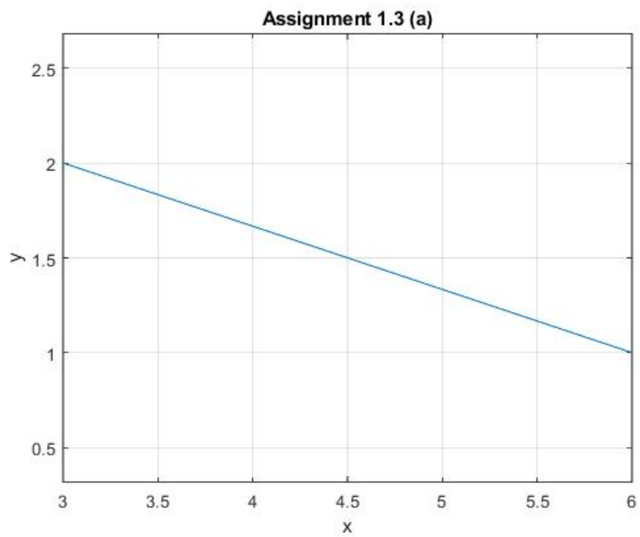

Figure 3: The resulting curve.

b) Given three points:

- $t_0 = 0, x_0 = (3, 2)$,
- $t_1 = 1, x_1 = (6, 1)$,
- $t_2 = 2, x_2 = (6, 6)$,
- $t_2 = 3, x_2 = (3, 10)$,

write down the interpolating polynomial. Show your work. Sketch the resulting curve (hint - plug in additional values of t to get the corresponding points).

Solution:
There are three points. Thus the degree k of the Lagrange polynomial curve is 3. That being said:

$$F(t) = \sum_{i=0}^{3} L_i(t) \cdot x_i, \tag{31}$$

And by expanding Eq. 26, we have

$$L_0(t) = \frac{t - t_1}{t_0 - t_1} \cdot \frac{t - t_2}{t_0 - t_2} \cdot \frac{t - t_3}{t_0 - t_3} = \frac{(t-1)(t-2)(t-3)}{-6} = \frac{t^3 - 6t^2 + 11t - 6}{-6} \tag{32}$$

$$L_1(t) = \frac{t - t_0}{t_1 - t_0} \cdot \frac{t - t_2}{t_1 - t_2} \cdot \frac{t - t_3}{t_1 - t_3} = \frac{t(t-2)(t-3)}{2} = \frac{t^3 - 5t^2 + 6t}{2} \tag{33}$$

$$L_2(t) = \frac{t-t_0}{t_2-t_0} \cdot \frac{t-t_1}{t_2-t_1} \cdot \frac{t-t_3}{t_2-t_3} = \frac{t(t-1)(t-3)}{-2} = \frac{t^3-4t^2+3t}{-2} \tag{34}$$

$$L_3(t) = \frac{t-t_0}{t_3-t_0} \cdot \frac{t-t_1}{t_3-t_1} \cdot \frac{t-t_2}{t_3-t_2} = \frac{t(t-1)(t-2)}{6} = \frac{t^3-3t^2+2t}{6} \tag{35}$$

Substitute the points into the equations above, yielding:

$$F(t) = \frac{t^3-6t^2+11t-6}{-6}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \frac{t^3-5t^2+6t}{2}\begin{pmatrix} 6 \\ 1 \end{pmatrix} \tag{36}$$

$$+ \frac{t^3-4t^2+3t}{-2}\begin{pmatrix} 6 \\ 6 \end{pmatrix} + \frac{t^3-3t^2+2t}{6}\begin{pmatrix} 3 \\ 10 \end{pmatrix} \tag{37}$$

$$= \begin{pmatrix} \frac{-3t^2+9t+6}{2} \\ \frac{-7t^3+39t^2-38t+12}{6} \end{pmatrix} \tag{38}$$

The figure is as follows:

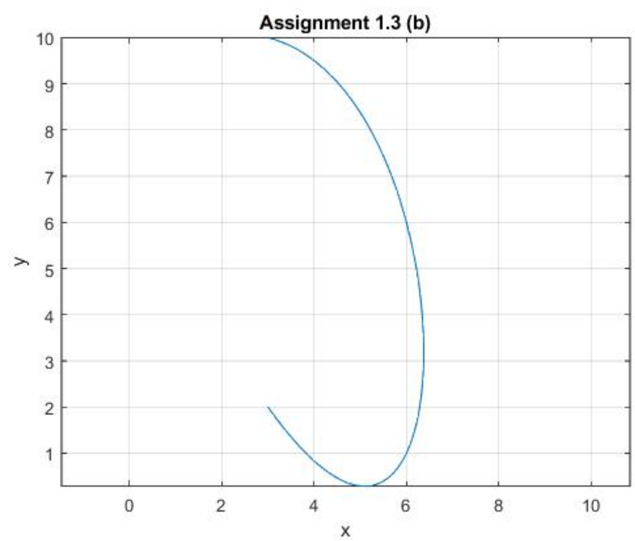

Figure 4: The resulting curve.