Assignment 4.1: Subdivision Curves (10 Points)

a) Two levels of subdivision of the initial rectangle. The arrows indicate how the first new position of the bottom-left vertex was constructed.

b) \[
\begin{bmatrix}
p^1_{2j-1} \\
p^1_{2j} \\
p^1_{2j+1}
\end{bmatrix}
= M
\begin{bmatrix}
p^0_{2j-1} \\
p^0_{2j} \\
p^0_{2j+1}
\end{bmatrix},
\] where 
\[
M = \begin{bmatrix}
1/2 & 1/2 & 0 \\
1/6 & 2/3 & 1/6 \\
0 & 1/2 & 1/2
\end{bmatrix}.
\]

c) Eigenvalues of \(M\) (computed using \texttt{eig()} command in Matlab, or by hand) are \(\lambda_0 = 1, \lambda_1 = 1/2\), and \(\lambda_2 = 1/6\). Because \(\lambda_0 = 1\) and \(|\lambda_2| < |\lambda_1| < |\lambda_0|\), we know that the subdivision scheme converges to a C1 continuous curve.
d) The (unnormalized) eigenvectors of $M$ are:

$x_0 = [1 \ 1 \ 1]^T; x_1 = [-1 \ 0 \ 1]^T; x_2 = [1 \ -2/3 \ 1]^T$

We want to express arbitrary vectors as linear combinations of the eigenvectors, i.e. $\vec{p} = \sum a_i x_i$. To find the coefficients $a_i$ (and therefore also the limit position and tangent), we use the rows of the inverse of the eigenvector matrix $X$:

$$X = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2/3 \\ 1 & 1 & 1 \end{bmatrix}; \quad X^{-1} = \begin{bmatrix} 2/10 & 6/10 & 2/10 \\ -5/10 & 0 & 5/10 \\ 3/10 & -6/10 & 3/10 \end{bmatrix}$$

From the first row of $X^{-1}$ we see that the limit position of a vertex is:

$$p^\infty_i = X_{00} \left( 2/10 p_{i-1} + 6/10 p_i + 2/10 p_{i+1} \right).$$

From the second row of $X^{-1}$ we see that the limit tangent at a vertex is:

$$t^\infty_i = -\frac{(5/10) p_{i-1} + (5/10) p_{i+1}}{\| (5/10) p_{i-1} + (5/10) p_{i+1} \|^2}.$$

We are not given the actual coordinates of the vertices of the control polygon, but using the above information we can speculate about the limit curve.

Let’s assume that the coordinates of the polygon are $(0, 0)$, $(2, 0)$, $(2, 1)$, and $(0, 1)$. Consider the bottom-left point at $(0, 0)$, which has predecessor $(0, 1)$ and successor $(2, 0)$.

By the above equations, the limit position is therefore $(4/10, 2/10)$, and the limit tangent is $(2, -1)/\|(2, -1)\|_2$.

Applying the same process to each vertex gives the following approximate picture:
a) Sketch a 2D curve with constant curvature. Explain your drawing.

A circle or an arc have a constant curvature equal to $\frac{1}{R}$, where $R$ is the radius. The parameterization of a circle by its arc-length is

$$F(t) = (R \cos\left(\frac{s}{R}\right), R \sin\left(\frac{s}{R}\right)).$$

Therefore, its curvature would be $||F''(t)||_2 = \frac{1}{R}$. Here is a picture of a circle:

![Circle](image)

b) Sketch a finite 2D curve which has regions of negative, positive, and zero curvature. Explain your drawing.

An example can be two arcs connected by a line, such as the one shown below. The linear part of the curve has a curvature of zero, while one of the arcs has a positive, and the other has a negative curvature.

![Finite 2D Curve](image)

c) Provide a real life example of a 3D curve with non-zero torsion. Explain your choice.

Any 3D curve which is not confined to a plane would have a non-zero torsion. An example would be a helix. Many springs, for example, have helical shapes. One way to parameterize a helix is $F(t) = (\cos(t), \sin(t), t)$.

![Helix](image)

d) Given the curve below, which starts at point A and ends at point B, has length $l$, and is parameterized using arc-length parameterization $s \in [0, l]$ mark the point on the curve at parameter value $s = l/3$. Explain your marking.

For an arc, the arc-length can be found as $s = R\theta$, where theta is the angle shown in the figure. Using this equation we find that $l = R\frac{3\pi}{4}$. Therefore, the arc-length $s = \frac{l}{3}$ would correspond to $\theta = \frac{\pi}{4}$.
Assignment 4.3: BONUS: Continuity of Bézier Curves (5 Bonus Points)

Assuming the conditions for $C_0$ continuity, we label the unknown Bézier control points $P_1$, $P_2$, $P_3$ and $P_4$ (see figure below).

We state first the continuity conditions in terms of the control points of the Bézier curves:

- $C_1$:
  - $(P_4 - C) = -(P_5 - C)$
  - $(P_1 - A) = -(P_6 - A)$
  - $(P_3 - B) = -(P_2 - B)$

- $C_2$:
  - $(C - P_5) + (P_6 - P_5) = -((C - P_4) + (P_3 - P_4))$
  - $(C - P_5) + (P_6 - P_5) = -((C - P_4) + (P_3 - P_4))$
  - $(C - P_5) + (P_6 - P_5) = -((C - P_4) + (P_3 - P_4))$

We make the following key observations:

Because of the $C_1$ continuity conditions and because the tangents to the curve at $A$, $B$ and $C$ are parallel to the opposite triangle edge we claim that the pairs of segments $((P_1, A), (P_6, A))$, $((P_2, B), (P_3, B))$, $((P_4, C), (P_5, C))$ have the same length and are parallel to $AB$, $BC$ and respectively $AC$ (see figure below)
Six unknown points in 2D amount to 12 variables. However, because of the orientation of the triangle and using the $C_0$ and $C_1$ continuity conditions we can reduce the number of variables to 3 as shown in the figure below.

We now use the $C_2$ continuity conditions listed above expressed in the variables $a$, $b$ and $c$ yielding the following system:

\[
\begin{align*}
\begin{pmatrix} -a \\ a/2 \end{pmatrix} - 2 \begin{pmatrix} -c \\ 1 \end{pmatrix} &= \begin{pmatrix} 2 \\ b \end{pmatrix} - 2 \begin{pmatrix} c \\ 1 \end{pmatrix} \\
\begin{pmatrix} c \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ b \end{pmatrix} &= \begin{pmatrix} a \\ -a/2 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ -b \end{pmatrix} \\
\begin{pmatrix} 2 \\ -b \end{pmatrix} - 2 \begin{pmatrix} a \\ -a/2 \end{pmatrix} &= \begin{pmatrix} -c \\ 1 \end{pmatrix} - 2 \begin{pmatrix} -a \\ a/2 \end{pmatrix}
\end{align*}
\]

Although the above system has 6 equations and only 3 unknowns, the equations are not linearly independent, hence it yields the following unique solution: $a = c = 2/3$, $b = 1/3$

The control points that we are looking for are shown in the following figures:
Note that we have also provided a more general answer for arbitrary number of input points in the Piazza post https://piazza.com/class/j9yuam8xj2c6o9?cid=199.