In this lecture we:

- CLIQUE is NP-Hard
- VERTEX COVER is NP-Complete
- 3-SAT is NP-Complete

## 1 CLIQUE ∈ NP-Hard

**Key idea:** reduce SAT to CLIQUE

**Steps:**
Transform a formula $\phi$ into a graph $G$ and integer $k$ so that
1. $G$ contains a clique of size $k \iff \phi$ is satisfiable.
2. Transformation is in polynomial time.

**Transformation Steps:**
1. Create a vertex for every literal in every clause.
2. Connect a vertex from $ith$ clause to vertex from $jth$ clause ($i \neq j$) unless they are negation of each other.
3. Let $k = \# \text{ clause in } \phi$

**Example:**

$\phi = (x_1) \land (\overline{x_1} \lor x_2) \land (x_1 \lor x_3) \land (x_2 \lor \overline{x_3} \lor x_4)$

$k = 4$
Claim: $\phi \in SAT$ iff $G$ has $k$-clique.

Proof:

$\Rightarrow$) If $\phi$ has a truth assignment then every clause has at least one true literal. If we choose the vertex corresponding to one true literal from each clause, these vertices form a clique of size $k = \text{the number of clauses}.$

$\Leftarrow$) If $G$ has a $k$-clique then exactly one vertex from each clause is in the clique. If we assign each literal corresponding to these vertices the value true, then every clause has a true literal and $\phi$ is satisfiable.

2 VERTEX COVER is NP-Complete

Vertex Cover Problem:
Given undirected graph $G = (V,E)$ and integer $k$. Does $G$ have a vertex cover of size $k$.

Vertex Cover:
Is a set vertices $S \subseteq V$ such that all edges in $G$ have at least one end point in $S$.

$$VC = \{<G,k> \mid G \text{ has a Vertex Cover of size } k\}$$

Theorem: $VC \in NPC$

Proof:
1. $VC \in NP$: certificate(witness) a $VC$ $S \subseteq V$ of size $k$. The verifier checks that $|S| = k$ and check for each $(u,v) \in E$ that $u \in S$ or $v \in S$.

2. CLIQUE $\rightarrow$ VC

$\overline{G}$ : the complement of $G : \overline{G} = (V,\overline{E})$ where $\overline{E} = \{(u,v) \mid (u,v) \notin E\}.$
Claim: $G$ has a clique of size $k \iff G' = \overline{G}$ has $VC$ of size $k' = |V| - k$

Proof:

$\Rightarrow$ if $G$ has a clique $S \subseteq V$, then $V - S$ is a $VC$ for $\overline{G}$. Consider if $(u,v) \in \overline{G}$, then $(u,v) \notin G$, which also implies either $u$ or $v$ is not in $S$ and either $u$ or $v$ is in $V - S$. Therefore, $(u,v)$ is covered.

$\Leftarrow$ Let $R \subseteq V$ be a vertex cover of $\overline{G}$ $\Rightarrow V - R$ is a clique of $G$.

$\Rightarrow$ If $(u,v) \in \overline{G}$, then $u$ or $v$ is in $R$.

$\Rightarrow$ if $u \notin R$ and $v \notin R$ then $(u,v) \notin \overline{G}$.

$\Rightarrow$ if $u \in V - R$ and $v \in V - R$ then $(u,v) \in G$.

3 3-SAT is NP-Complete

Theorem: 3 – SAT $\in$ NPC
3 – SAT = { $\phi$ | each clause has at most 3 literals and $\phi$ is in SAT}
idea of transform: \((a_1 \lor a_2 \ldots \lor a_k) \Rightarrow (a_1 \lor a_2 \lor y_1)(\overline{y_1} \lor a_3 \lor y_2)\ldots(\overline{y_{k-3}} \lor a_{k-1} \lor a_k)\)

**Claim:** \(\phi\) is satisfiable \iff \(\phi'\) is satisfiable.

**Proof:**

\(\Rightarrow\) If \(\phi\) is satisfiable, then there is a truth assignment so that each clause has a true literal. Let \(a_i\) be a true literal for clause \(C = (a_1 \lor a_2 \lor \cdots \lor a_k)\), then set \(y_1, y_2, \ldots, y_{i-2}\) to true and \(y_{i-1}, y_i, \ldots, y_{k-3}\) to false. In this way, every one of the 3-SAT clauses derived from \(C\) is satisfied.

\(\Leftarrow\) If \(\phi'\) is satisfiable, then every one of the 3-SAT clauses derived from \(C = (a_1 \lor a_2 \lor \cdots \lor a_k)\) has a true literal. At least one of the \(a_i\)'s must be true. Otherwise, if all \(a_i\)'s are false, \(y_1\) must be true (to satisfy the first 3-SAT clause) which implies \(y_2\) must be true (to satisfy the second 3-SAT clause) which implies, eventually, that \(y_{k-3}\) must be true which implies that the last 3-SAT clause is false: a contradiction. Since at least one of the \(a_i\)'s is true, the clause \(C\) is satisfied.