EXAMPLE: Salad Dressing

<table>
<thead>
<tr>
<th>Bottle</th>
<th>Oil</th>
<th>Vinegar</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>15%</td>
<td>36%</td>
</tr>
<tr>
<td>B</td>
<td>9%</td>
<td>21%</td>
</tr>
</tbody>
</table>

- Can we mix bottles to get 13% of oil and 31% of vinegar?
  -> Yes! 2 Parts A and 1 part B
  
  Oil: \( \frac{2}{3} (15\%) + \frac{1}{3} (9\%) = 13\% \)
  
  Vinegar: \( \frac{2}{3} (36\%) + \frac{1}{3} (21\%) = 31\% \)

- Can we mix bottles to get 12% of oil and 30% vinegar?
  -> No :(

Referring to the figure above, the black line segment reflects the solutions obtainable by mixing bottles A and B.

If another bottle C of 12% oil and 33% vinegar is added (as shown by the orange line segment), then we can not only obtain a mixture of 12% oil and 30% vinegar, but any points enclosed within the triangle formed by A, B, and C.

A mixture is a convex combination of points representing the contents of the bottles, where points \( p_1, p_2, \ldots, p_n \) is \( \sum_{i=1}^{n} \alpha_i p_i \) such that

- \( \sum_{i=1}^{n} \alpha_i = 1 \)
- \( \alpha_i \geq 0 \) for all \( i \)
Convex Hulls (CH(P))

Convex hull (CH(P)) of a set of points P is the smallest convex set containing P, the set of all convex combinations of P.

A set T is convex if for all a, b ∈ T, segment ab is in T.

The boundary of the convex combination are bounded by the perimeter points of the set.

Problem

Input: set of points \( P = p_1 p_2 \cdots p_n \)

Output: convex hull of P

Approach 1: Jarvis March ("Gift-wrapping") 1973

Gist of Algorithm

Jarvis march

- Start at the leftmost point in the set (if there are multiple, take the closest to the bottom)
- For all other points, compute the angle measured ccw with down being 0°
- Take point with smallest angle
- Repeat, except for further steps, measure angles ccw with the line segment from current to previously selected point being 0°

Psuedocode from Class

1. Find point \( p_0 \) in P with minimum y-coordinate ---- O(n)
2. set \( h = 0 \) ---- O(1)
3. repeat for each \( p \in P \)
   - if right turn \((p_h, q, p)\)
     - \( q = p \)
     - \( p_{h+1} = q \)
     - \( h = h + 1 \)
   until \( p_h = p_0 \)

Algorithm Jarvis' March [1973]

- gift-wrapping method, generalizes to higher dimensions
- Step 1: Let \( p_0 \) be the point with minimum y-coordinate (lex.)
- Step 2: Anchor ray at current point and rotate to next anchor point. Repeat.

Output-sensitive: O(nh) time.

- n = # input points, h = # hull vertices (output size)
- (3 ≤ h ≤ n if n ≥ 3 and not all points collinear)
- Worst-case: O(n^2) time.
**Approach 2: Graham's Scan (1972)**

**GRAHAM-SCAN(Q)**

1. let $p_0$ be the point in $Q$ with the minimum $y$-coordinate, or the lefthmost such point in case of a tie
2. let $(p_1, p_2, \ldots, p_n)$ be the remaining points in $Q$, sorted by polar angle in counterclockwise order around $p_0$ (if more than one point has the same angle, remove all but the one that is farthest from $p_0$)
3. let $S$ be an empty stack
4. PUSH($p_0$, $S$)
5. PUSH($p_1$, $S$)
6. PUSH($p_2$, $S$)
7. for $i = 3$ to $n$
8. while the angle formed by points NEXT-TO-TOP($S$), TOP($S$), and $p_i$ makes a nonleft turn
9. POP($S$)
10. PUSH($p_i$, $S$)
11. return $S$