CS420+500: Advanced Algorithm Design and Analysis

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Lecture Apr 3rd & 5th

Note Summary: Universal Hash functions & Corresponding properties

Cuckoo Hashing & Its properties

Universal Hash Functions:

Theorem:

For key k, $E[n_{h(k)}] \leq \begin{cases} \alpha + 1 \; if \; k \in T \\ \alpha \; otherwise \end{cases}$

Notations: n_i = number of items in bucket i of hash table T

n = total number of items in T

m = size of T

$\frac{n}{m} = load\ factor\ of\ T = \alpha$

Proof:

Let X_{kl} = $\begin{cases} 1 \; if \; h(k) = h(l) \\ 0 \; otherwise \end{cases}$ where h is the hash function used

Let Y_k = number of keys that hash to same slot as k (excluding k)

Then $Y_k = \sum_{l \neq k}^{l \in T} X_{kl}$

$E[Y_k] = E[\sum_{l \neq k}^{l \in T} X_{kl}] = \sum_{l \neq k}^{l \in T} E[X_{kl}] = \sum_{l \neq k}^{l \in T} \frac{1}{m}$

If $k \notin T$ then $n_{h(k)} = Y_k$ and $|\{l \in T | \; l \neq k\}| = n \; \rightarrow E[Y_k] = \frac{n}{m}$

On the other hand, if $k \in T$ then $n_{h(k)} = Y_k + 1$ and $|\{l \in T | \; l \neq k\}| = n - 1 \; \rightarrow$ $E[Y_k] = \frac{n-1}{m}$

See example on the next page.

Example of universal set of hash functions:

$h_{a,b}(k) = \big((ak + b)\ mod\ p\big)\ mod\ m$

"p" is a prime number greater than any key, "p" is fixed

Choose $\begin{cases} a \in \{1,2, \dots \dots p-1\} \\ b \in \{0,1, \dots \dots p-1\} \end{cases}$ to select hash function of set

Cuckoo Hashing:

More detailed info at: https://en.wikipedia.org/wiki/Cuckoo_hashing

Notes:

Runtime of cuckoo hashing, find takes O(1), delete takes O(1), insert takes O(1) but amortized expected.

How it works: Use 2 hash functions h_1 and h_2, key k will either be stored at $h_1(k)\ or\ h_2(k)$, each slot in table T contains one key.

Pseudo Code for cuckoo hashing:

Insert(k)

pos = $h_1(k)$

repeat n times

{

b = T[pos]

T[pos] = k

If b == NULL, return ← $cuckoo\ k\ has\ kicked\ out\ b$

If pos = $h(b)\ then\ pos = h_2(b)$

Else pos = $h_1(b)$

k = b ← $b\ becomes\ new\ cuckoo$

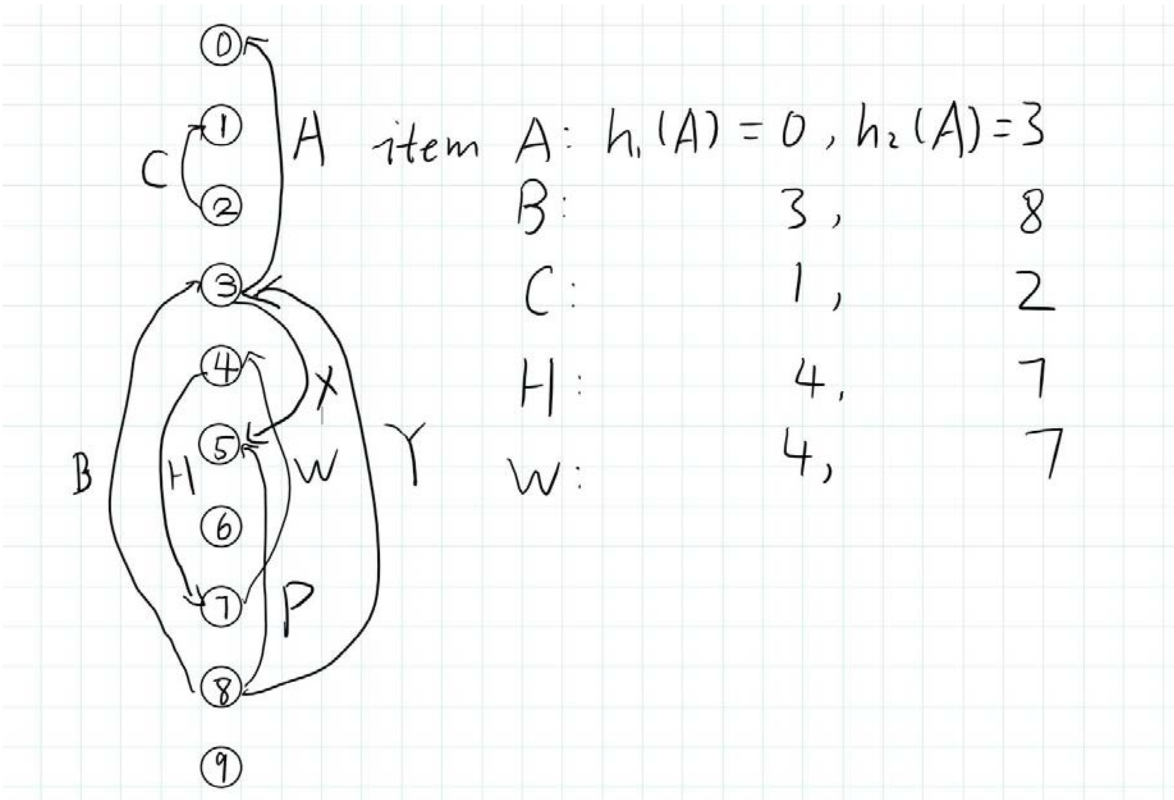
}

Rehash() and Insert(k)

Cuckoo Graph for n items:

Vertices = {0,1,…..,m-1}

Edges = {($h_1(k), h_2(k)$)| k∈ T}



Assume $h_1(k)$ and $h_2(k)$ $are\ uniform\ and\ independent\ random\ edges$, an insertion will succeed if there is no cycle in the cuckoo graph.

Lemma: For $c > 1$ and $m \geq 2cn$

$$\Pr[cuckoo\ graph\ has\ path\ of\ length\ l\ from\ i\ to\ j] \leq \frac{1}{mc^l}$$

“c” is a constant less or equal to 3

Proof by induction on l:

Base: l = 1 Edge(i,j) exists in graph with Pr $\leq n\frac{2}{m^2} = n\frac{2}{2cnm} = \frac{1}{c^1 m}$

Note: $\Pr[a\ single\ random\ edge\ being(i,j)] = \frac{2}{m^2}$

When $l > 1$ shortest path from i to j has length l only if there exists p and

$$\begin{cases} 1, \text{there exists a shortest path from } i \text{ to } p \text{ of length } l-1 \text{ with } pr \leq \dfrac{1}{mc^l} \\ 2, \text{there exists the edge } (p,j) \text{ with } pr \leq \dfrac{1}{mc} \end{cases}$$

Together, $\leq \frac{1}{m^2c^l}$ $sum\ over\ possible\ p\ \rightarrow \leq \frac{1}{mc^l}$

Probability that k and k' hash to the same path/bucket is probability of a path from

$$h_1(k) or\ h_2(k)\ to\ h_1(k') or\ h_2(k') \leq 4 \sum_{l=1}^{\infty} \frac{1}{mc^l} = \frac{4}{m} \frac{1}{c-1} = O(\frac{1}{m})$$

Rehash means choosing new hash functions and rehashing all keys, Pr[rehash]

$\leq Prob\ hashing\ creates\ cuckoo\ graph\ with\ a\ cycle$

$$\leq \sum_{l=i}^{m} \Pr[cycle\ involving\ i] \leq \sum_{i=1}^{m} \sum_{l=1}^{\infty} \frac{1}{mc^l} \leq \frac{1}{2}\ for\ c \geq 3$$