Mar.22+Mar.24

Euclidean TSP is NP-hard [Papadimitriou, 1977]

Hamiltonian Cycle Problem: Given an unweighted graph G, does G contain a cycle that visits every vertex exactly once?

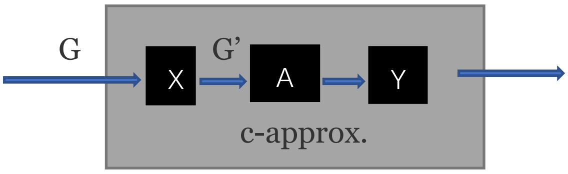
Hardness of Approximation

The general TSP is NP-hard to approximate

Claim: If $P \neq NP$, then there is no polynomial time c-approximation algorithm for TSP.

Proof: Suppose A is a polynomial time c-approximate algorithm for TSP, we use A to solve Hamiltonian cycle problem.

Hamiltonian Cycle problem



Transform X:

Create G' from G=<V, E> such that G' has all edges. Assign weights to edges in G' as follows:

$$w(u, v) = \begin{cases} 1 \text{ if } (u,v) \in G \\ c * |V| + 1 \text{ if } (u,v) \notin G \end{cases}$$

(i.e. using a non-existing edge is (much) worse than using an existing one)

Transform Y:

If $|TSP(G')| \leq c*|V|$ then output "Yes", return "No" otherwise.

Correctness of reduction:

Edges not in the original graph are so costly that there is a gap between cost of tour if G contains a Hamiltonian cycle (cost = u) and cost of tour if G doesn't (cost > $c*|V|$)

New topic: Online Algorithms

For input sequences $p_1p_2\ldots p_n$(very large), an online algorithm must produce an output given $p_1p_2\ldots p_i$(without seeing $p_{i+1}\ldots p_n$) for each i.

Example:

Page replacement in cache

$P_1p_2\ldots p_n$ is a sequence of page requests made by a program, k is a cache (number of pages). At i-th page request, p_i, the cache contains some k pages. If p_i is not in cache (page fault), some page must be evicted from cache to make room for pi then p_i is added to cache. The cost of a page replacement algorithm A on a sequence $p_1p_2p_3...p_n$ is $f_A(p_1p_2p_3..p_n)$ = the number of faults on $p_1p_2\ldots p_n$

Online algorithm must decide what page to evict without knowing the future request.

Examples of page replacement algorithms:

1. Least Recently Used (LRU):
   Evict page whose most recent request occurred furthest in the past (the least recently used page).

2. Least Frequently Used (LFU):
   Evict page that has been requested least often.

3. Marking Algorithms: “poor man’s LRU” with randomization.

4. FIFO (First In First Out):
   Evict page that has been in cache longest.

How do we decide the “best” online algorithm?

1. Worst case performance

   $$\max\begin{cases} f_{LRU}(p_1p_2\ldots p_n) = n \\ f_{LFU}(p_1p_2\ldots p_n) = n \\ f_{FIFO}(p_1p_2\ldots p_n) = n \end{cases}$$

2. Average case performance
   m = total number of pages possibly requested. Expected number of page faults on sequence of randomly, uniformly, independently chosen pages:
   $E[f_{\substack{LFU\\LRU\\FIFO}}(p_1p_2\ldots p_n)] = (1-k/m)*n$
3. Competitive Analysis
   How does online algorithm’s performance compare to best offline algorithm?
   Definition: An online algorithm A is c-competitive if exist b such that for all $p_1p_2\ldots p_n$:
   $f_A(p_1p_2\ldots p_n) \le c*f_{OPT}(p_1p_2\ldots p_n) + b$ where b is an arbitrary constant
   Theorem: LRU and FIFO are k-competitive where k is the cache size.
   Theorem: If A is a deterministic online algorithm for paging, then c>=k.