CS420+500: Advanced Algorithm Design and Analysis

Lectures: March 20 + March 22, 2017

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In this lecture we discussed:

- **Traveling Saleman Problem (TSP)**
- **Triangle TSP** (Δ TSP)
- **Algorithm for** Δ TSP

Handouts (posted on webpage):

- HANDOUT NAME: None

Reading: Probably want to check http://jeffe.cs.illinois.edu/teaching/algorithms/notes/31-approx.pdf.

1 Approximate Algorithms: Traveling Saleman Problem (TSP)

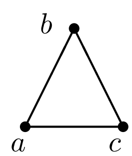
- Traveling Saleman Problem (TSP)

  definition: Given a graph with weight (distances) on its edges, you must find a cycle of minimum total weight that visits each vertex exactly once.

- Δ TSP

  This is similar to TSP except that Triangle TSP has edges weights obeying triangle inequality.

  The triangle inequality rule is illustrated in the following triangle



  $d(a, b) \leq d(a, c) + d(c, b)$ where d is distance

  We will be discussing Δ TSP for the rest of this lecture.

- **Christofides Algorithm for triangle TSP (1976)**

  The algorithm is as below:

  1. Find a minimum spanning tree (MST) of G. call this T = MST

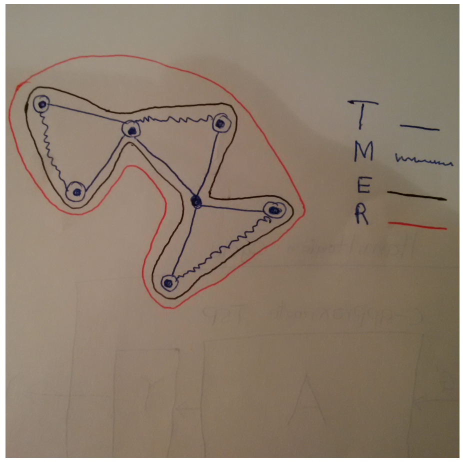


Figure 1: T (tree) matching to demonstrate TSP

2. Compute minimum length complete (perfect) matching M in the complete graph on odd-degree vertices of T (tree) T
3. Find Euler tour E in $T \cup M$
   - we define Euler tour as a cycle that visits every edge exactly once.
4. Eliminate repeated vertices from E to get TSP tour R

- **Things to note**
  1. A perfect matching always exists in step 2 above because there are an even number of odd degree vertices
  2. Euler tour exist because every vertex in $T \cup M$ has even-degree.
  3. A TSP tour minus one edge is a spanning tree: which implies that:
     $|T| \leq |TSP(G)|$ where $|T|$ is the total edge weight in $|T|$

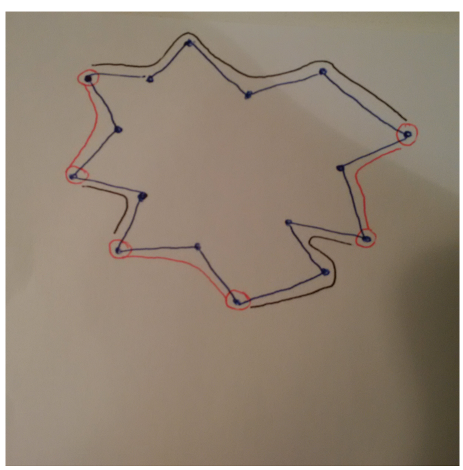


Figure 2: Two Odd-degrees matching; black as B and red as D

4. – figure 2 above shows two matching for odd degree vertices

– call the black odd-degree matching B and the red odd-degree matching D then we have
– $|B| + |D| \leq |TSP(G)$ therefore,
– $|M| \leq |B|$ and $|M| \leq |D|$ because M is minimum weight matching which implies that: $|M| \leq \frac{1}{2}|TSP(G)|$
– Euler tour E has $|E| \leq \frac{3}{2}|TSP(G)|$
– And the final approximation tour is: $|R| \leq |E| \leq \frac{3}{2}|TSP(G)|$

2 Euclidean TSP is NP-Hard (Papadimintrion 1977)

Hamiltonian Cycle

Given unweighted graph G, does G contain a cycle that visits every vertex once? That is the basis of Hamiltonian cycle.

What if we want to approximate the general TSP? Ans. No.

- **Hardness of Approximation**

The general TSP is NP-Hard to approximate

`Claim:` If $P \neq NP$ `then there is no polynomial time c-approximation algorithm for TSP`

Proof

Suppose A is polynomial time c-approximation algorithm for TSP we use A to solve Hamiltonian cycle

Transform X: create G' from G = (V,E).

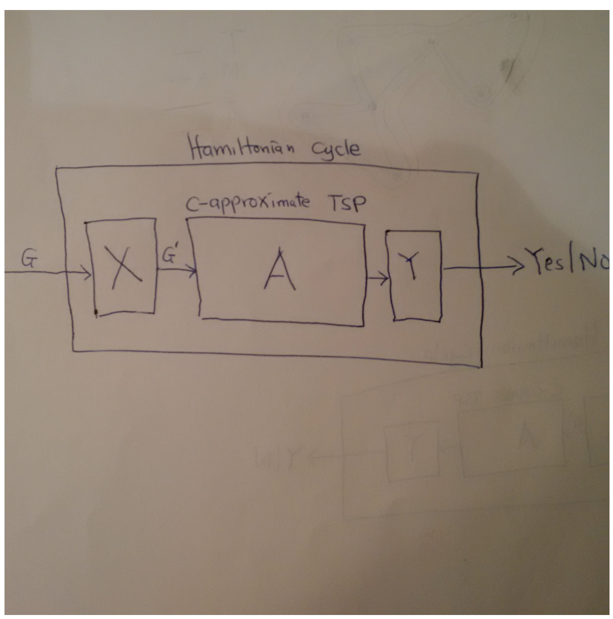


Figure 3: Hamiltonian Cycle and Transformation

G' has all edges

$$w(u,v) = \begin{cases} 1 & \text{if } (u,v) \in G \\ \\ |V|+1 & if\text{(u,v)} \notin G \end{cases}$$

Transform Y: $|TSP_A(G')| \leq c|V|$ then output yes otherwise output no

Why does it work?

Edges not in the original graph are so costly that there is a gap between cost of tour if G contains a Hamiltonian cycle (cost = n) and cost of tour if G does not ($cost > c|V|$)

3 NEXT TOPIC

- **On-Line Algorithms**