CS420+500: Advanced Algorithm Design and Analysis

Lectures: Mar 8 + Mar 10, 2017

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In this lecture we:

- CLIQUE is NP-Hard
- VERTEX COVER is NP-Complete
- 3-SAT is NP-Complete

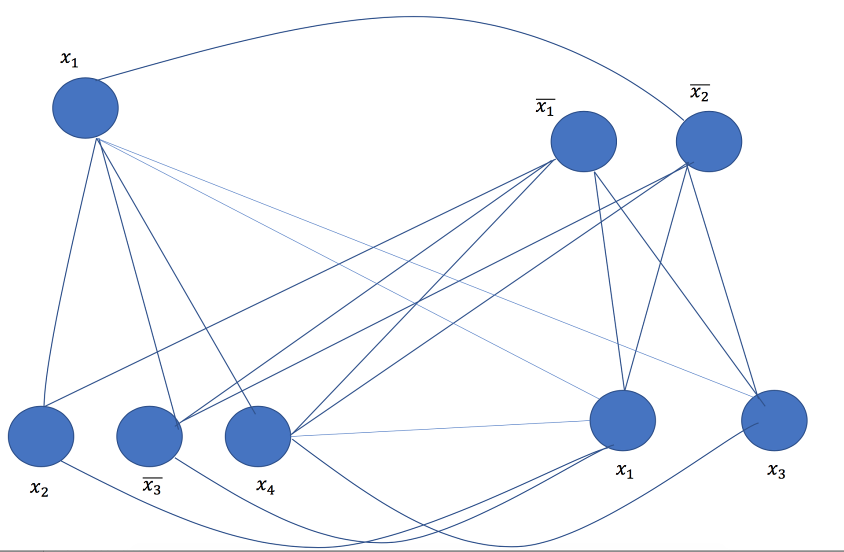
1 CLIQUE $\in$ NP-Hard

Key idea: reduce SAT to CLIQUE

Steps:
Transform a formula ϕ into a graph G and integer k so that
1. G contains a clique of size $k \iff \phi$ is satisfiable.
2. Transformation Is in polynomial time.

Transformation Steps:
1. Create a vertex for every literal in every clause.
2. Connect a vertex from ith clause to vertex from jth clause ($i \neq j$) unless they are negation of each other.
3. Let $k = \#$ clause in ϕ

Example:
$\phi = (x_1) \wedge (\overline{x_1} \vee \overline{x_2}) \wedge (x_1 \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee x_4)$
$k = 4$



Claim: $\phi \in SAT$ iff G has k-clique.

Proof:

$\Rightarrow$) If ϕ has a truth assignment then every clause has at least one true literal.
If we choose the vertex corresponding to one true literal from each clause, these vertices form a clique of size $k =$ the number of clauses.

$\Leftarrow$) If G has a k-clique then exactly one vertex from each clause is in the clique.
If we assign each literal corresponding to these vertices the value true, then every clause has a true literal and ϕ is satisfiable.

2 VERTEX COVER is NP-Complete

Vertex Cover Problem:
Given undirected graph $G = (V, E)$ and integer k. Does G have a vertex cover of size k.

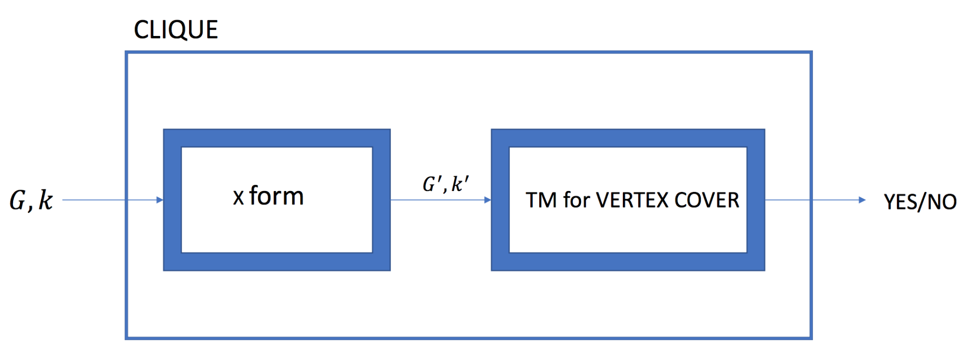
Vertex Cover:
Is a set vertices $S \subseteq V$ such that all edges in G have at least one end point in S.

$$VC = \{< G, k > \mid G \ has \ a \ Vertex \ Cover \ of \ size \ k\}$$

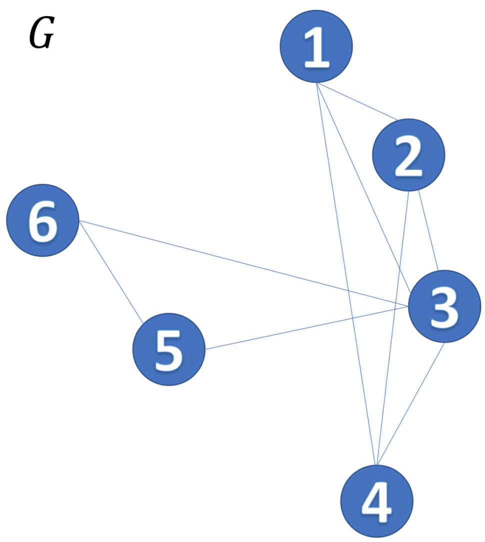
Theorem: $VC \in NPC$

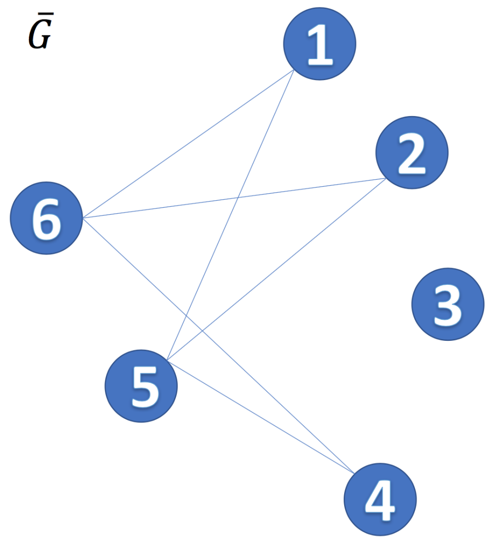
Proof:
1. $VC \in NP$: certificate(witness) a VC $S \subseteq V$ of size k. The verifier checks that $|S| = k$ and check for each $(u, v) \in E$ that $u \in S$ or $v \in S$.

2. CLIQUE $\longrightarrow VC$



$\overline{G}$: the complement of $G : \overline{G} = (V, \overline{E})$ where $\overline{E} = \{(u, v) | (u, v) \notin E\}$.





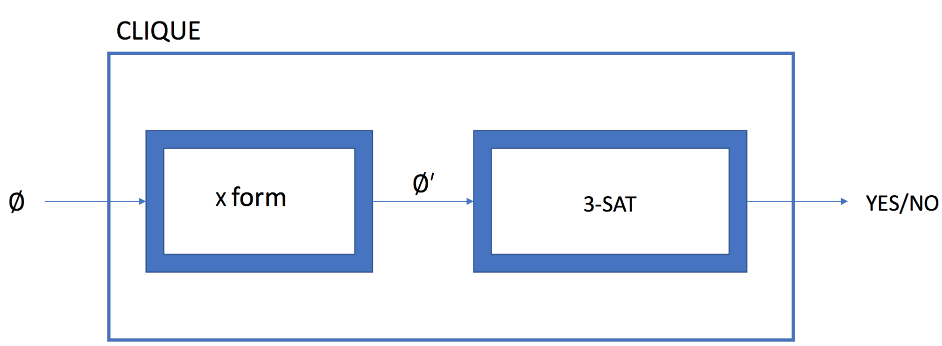
Claim: G has a clique of size k $\iff$ $G' = \overline{G}$ has VC of size $k' = |V| - k$

Proof:
$\Rightarrow$) if G has a clique $S \in V$, then $V - S$ is a VC for $\overline{G}$. Consider if $(u, v) \in \overline{G}$, then $(u, v) \notin G$, which also implies either u or v is not in S and either u or v is in $V - S$. Therefore, (u, v) is covered.

$\Leftarrow$) Let $R \subseteq V$ be a vertex cover of $\overline{G} \Rightarrow V - R$ is a clique of G.
$\Rightarrow$ If $(u, v) \in \overline{G}$, then u or v is in R.
$\Rightarrow$ if $u \notin R$ and $v \notin R$ then $(u, v) \notin \overline{G}$.
$\Rightarrow$ if $u \in V - R$ and $v \in V - R$ then $(u, v) \in G$.

3 3-SAT is NP-Complete

Theorem: $3 - SAT \in NPC$
$3 - SAT = \{\ \phi \mid each\ clause\ has\ at\ most\ 3\ literals\ and\ \phi\ is\ in\ SAT\}$



idea of transform: $(a_1 \vee a_2 ... \vee a_k) \Rightarrow (a_1 \vee a_2 \vee y_1)(\overline{y_1} \vee a_3 \vee y_2)...(\overline{y_{k-3}} \vee a_{k-1} \vee a_k)$

Claim: ϕ is satisfiable $\iff$ $\phi^{'}$ is satisfiable.

Proof:

$\Rightarrow$)If ϕ is satisfiable, then there is a truth assignment so that each clause has a true literal. Let a_i be a true literal for clause $C = (a_1 \vee a_2 \vee \cdots \vee a_k)$, then set $y_1, y_2, \ldots, y_{i-2}$ to true and $y_{i-1}, y_i, \ldots, y_{k-3}$ to false. In this way, every one of the 3-SAT clauses derived from C is satisfied.

$\Leftarrow$)If ϕ' is satisfiable, then every one of the 3-SAT clauses derived from $C = (a_1 \vee a_2 \vee \cdots \vee a_k)$ has a true literal. At least one of the a_i's must be true. Otherwise, if all a_i's are false, y_1 must be true (to satisfy the first 3-SAT clause) which implies y_2 must be true (to satisfy the second 3-SAT clause) which implies, eventually, that y_{k-3} must be true which implies that the last 3-SAT clause is false: a contradiction. Since at least one of the a_i's is true, the clause C is satisfied.