CPSC 420/500 Scribe Notes
Lectures: Feb. 15 and Feb. 17

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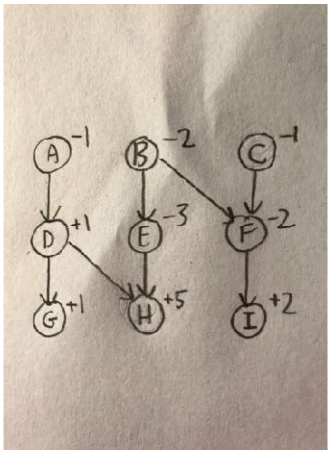
Open Pit Mining - A Network Flow Example

Continuing from last class, we were working with a directed acyclic graph as follows:



Then add a source vertex s, and sink vertex t. For each vertex v with negative weight w(v), place an edge from s to v with weight equal |w(v)|. For each vertex u with positive weight w(u), place an edge from u to t with weight equal |w(u)|. This allows us to find the max profit.

In terms of the minimal cut, the cut dividing the graph into the source and sink sides is:
$(S, T) = (\{s, C, F, I\}, \{A, B, D, E, H, G, t\})$

The overall profit works out to be +1, and the sum of the positive weights on the edges going to t are $\sum_{u \in U, w(u)>0} w(u) = 9$.

Now:

maximize profit = $\sum_{u \in U, w(u)>0} w(u)$ + $\sum_{v \in V, w(v)<0} w(v)$ otherwise written as:

minimize -profit = $-\sum_{u \in U, w(u)>0} w(u)$ + $-\sum_{v \in V, w(v)<0} w(v)$

The capacity of the cut associated with the initial set U, where $T = U \cup \{t\}$ and $S = V - T$ is:

cap(S,T) = $\sum_{u \notin U, w(u)>0} w(u) + -\sum_{v \in V, w(v)<0} w(v)$

and thus: $\sum_{v \in V, w(v)>0} w(v)$ - profit = $\sum_{u \notin U, w(u)>0} w(u) + -\sum_{v \in V, w(v)<0} w(v)$

Note that the $\sum_{v \in V, w(v)>0} w(v)$ term is independent of the set U, and all the terms in this summation are positive values. Since this term is independent of U, minimizing the capacity over the various possible cuts (or over the possible initial sets) is equivalent to maximizing profit.

Linear Programming Duality

Given the following linear program:

$\max x_1 + 6x_2$

such that:

$$\begin{aligned} x_1 \quad\quad &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$(x_1, x_2) = (100, 300)$ is the optimal solution to the LP.

The value of the optimal solution is 1900.

Multiplying the 1st inequality by 0, the 2nd by 5, and the third by 1, and summing them, gives a bound on the value of the optimal solution as follows:

$$\begin{aligned} 0 * x_1 \quad\quad &\leq 0 * 200 \\ 5 * x_2 &\leq 5 * 300 \\ 1 * x_1 + 1 * x_2 &\leq 1 * 400 \end{aligned}$$

and summing them gives:

$x_1 + 6x_2 \leq 1900$

as required.

The multipliers above are the optimal solution to the dual linear program. These multipliers must be non-negative. Thus the optimal solution to the dual is $(y_1, y_2, y_3) = (0, 5, 1)$.

We can write $(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$

Now $x_1 + 6x_2 \leq 200y_1 + 300y_2 + 400y_3$ as long as $(y_1 + y_3) \geq 1$ and $(y_2 + y_3) \geq 6$.

We can now write the dual LP as follows:

$\min\ 200y_1 + 300y_2 + 400y_3$

such that:

$$
\begin{aligned}
y_1 + \quad\quad\quad\quad y_3 &\geq 1 \\
y_2 + \quad y_3 &\geq 6 \\
y_1, y_2, y_3 &\geq 0
\end{aligned}
$$

Duality Theorem

If an LP has a bounded optimum, then so does its dual, and the optimum values are the same.

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Let $I \subseteq \{1, \dots, m\}$ and $N \subseteq \{1, \dots, n\}$

Then the Primal LP and the Dual LP will have the following form:

Primal:

$\max c_1x_1 + \dots + c_nx_n$

such that:

$$
\begin{aligned}
a_{i1}x_1 + \dots + a_{in}x_n &\leq b_i \ \ for\ all\ i \in I \\
a_{i1}x_1 + \dots + a_{in}x_n &= b_i \ \ for\ all\ i \notin I \\
x_j &\geq 0 \ \ for\ all\ j \in N
\end{aligned}
$$

Dual:

$\min b_1y_1 + \dots + b_my_m$

such that:

$$
\begin{aligned}
a_{1j}y_1 + \dots + a_{mj}y_m &\geq c_j \ \ for\ all\ j \in N \\
a_{1j}y_1 + \dots + a_{mj}y_m &= c_j \ \ for\ all\ j \notin N \\
y_i &\geq 0 \ \ for\ all\ i \in I
\end{aligned}
$$

Two Player Zero-Sum Games

One player's gain is the other's loss in this type of game.

Consider the game Rock Paper Scissors with the following payoff matrix:

Column

Row

	r	p	s
r	0	-1	1
p	1	0	-1
s	-1	1	0

The amount shown is the amount the Row player receives, or the column player pays.

For example, if the Row player plays "r" every time, then the column player can play "p" and win every time.

We introduce the concept of a mixed strategy to solve this problem. A mixed strategy is a probability distribution on the actions (x_1, x_2, x_3) for the Row Player and (y_1, y_2, y_3) for the Column player, in the case of the game just mentioned.

Row player's strategy: (x_1, x_2, x_3)

Column player's strategy: (y_1, y_2, y_3)

Expected Payoff is $\sum_{ij} G_{ij} \Pr[Row\ plays\ i\ and\ Column\ plays\ j]$

= $\sum_{ij} G_{ij}\, x_i y_j$ where G_{ij} corresponds to the ij entry of the Payoff Matrix, x_i corresponds to the ith entry of the Row player's strategy, and y_j corresponds to the jth entry of the Column player's strategy.

The Row player wants to maximize the expected payoff, where as the Column player wants to minimize the expected payoff.

In the case of Rock Paper Scissors, if the Row player plays the strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, the expected payoff is 0 for the Row player. Similarly, if the Column player plays the same strategy, the expected payoff for them is also 0. It turns out that both the Row and Column players cannot play a different strategy and hope for a better expected payoff. This game is an example of a fair game.

Here is another example of a 2 player zero sum game:

Column

Row

3	-1
-2	1

If the Row player plays $(\frac{1}{2}, \frac{1}{2})$ then:

If the column player plays $(1{,}0)$ then column pays $\frac{1}{2}$. Otherwise if column player plays $(0{,}1)$ then column pays 0.

In general, for any row strategy, there is a pure optimal column strategy.

Now consider:

If the Row player plays first with (x_1, x_2) then the Column player can achieve min$\{3x_1 - 2x_2,\ 3x_1 - 2x_2\}$. Row picks (x_1, x_2) to maximize this.

The following LP will give the optimal strategy for the Row player:

$\max z$

such that:

$$z \leq 3x_1 - 2x_2$$
$$z \leq -x_1 + x_2$$
$$x_1 + x_2 = 1$$
$$x_1, x_2 \geq 0$$

If the Column player plays first with (y_1, y_2) then the Row player can achieve

$\max\{3y_1 - y_2, -2y_1 + y_2\}$. Column picks (y_1, y_2) to minimize this.

The following LP will give the optimal strategy for the Column player:

$\min w$

such that:

$$
\begin{aligned}
w &\leq 3y_1 - y_2 \\
w &\leq -2y_1 + y_2 \\
y_1 + y_2 &= 1 \\
y_1, y_2 &\geq 0
\end{aligned}
$$