CPSC 420+500: Advanced Algorithm Design and Analysis
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Applications of Network Flows

Last time: Max matching in bipartite graph is defined by the maximum flow.

Proof: Max flow will give a maximum matching.

Consider the main property of a bipartite graph: vertices in one partition have no connections between them. That allows us to set up "disjoint" unique paths from s to t. Where "disjoint" means that no two paths go through same vertex in either partition. For example, by having a unit flow on the s -> **v** -> u ->t path, we can't have flow on the s -> **v** -> w -> t path. On this property we'll base our proof.

Claim:

1) If M is a matching in a bipartite graph G, then there exists a flow f in F (flow network for G) such that $size(f) = M$. In other words, we can write:
   $size(f^*) \geq |M^*|$, where f^* is a max flow and M^* is a size of maximum matching set.
2) If f is an integer value flow in F, then there exists a matching M in G of size $size(f)$.
   $size(f^*) \leq |M^*|$

For the claim # 2, let $M = \{(u, v) \in G \mid f(u, v) = 1\}$ then M forms a matching.

Why? Because all capacities of the edges going out of s (source) are equal to 1 and all capacities of the edges coming to t (sink) are equal to 1. Therefore, we have both inequalities together, which defines: $size(f^*) = |M^*|$ maximum flow equals to maximum matching.

Pennant Race Problem (1965)

The Pennant Race problem is:

Input: Given a baseball team A, a list of any other teams $T_1, T_2, \ldots, T_n$ to win the pennant against team A. There is a win/loss rate for all teams which already played the games.

Question: Is it possible that team A can end the season having won at least as many games as any other team?

Conditions:
- Since it's baseball, each game has winner and loser, no draw.
- All schedule games are played.

Initial set up:

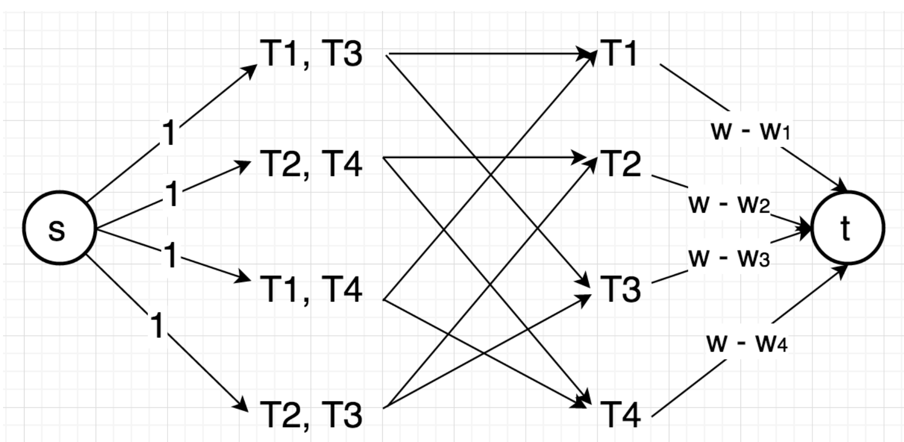
Consider a table where each team has # of wins over the played games, potential winning points from the future games, and the set of the remaining games. For *A* we can calculate the potential winning points easily, just assume *A* won all the remaining games.

Team	# of wins	Potential win points (w)	Remaining games
A	3	6	(A, T_1), (A, T_3), (A, T_4), (T_1, T_3), (T_2, T_3), (T_2, T_4), (T_1, T_4)
T_1	$4 = w_1$		
T_2	$6 = w_1$		
T_3	$5 = w_1$		
T_4	$4 = w_1$		

Under assumption that *A* will win all the remaining games:

- Let w = # wins after *A* wins all the remaining games;
- Let w_i = # wins for T_i (current);
- If $w < w_i$ for some i, then *A* has no hope. Otherwise, consider the set of games remaining to be played $\{(T_i, T_j)\}$ where $i \neq j$.

We can transform the problem into a Bipartite graph. The idea is to think about wins as of 1 unit flows. Thus each game will have one "flow" = "win" to a team. Assume, $w > w_i$ for all i, such that we still have a hope before we run an algorithm.



Edges:

(s, T_i, T_j) with capacity 1;
(T_i, T_j, T_i) any capacity ≥ 1;
(T_i, t) with capacity $w - w_i$, the maximum number of games T_i can win without getting more wins than team *A*.

After constructing the graph, we run any algorithm to find the max flow of the bipartite graph. If *size(max flow)* is less than the number of games remaining to be played (after assuming team *A* wins all its remaining games), then team *A* has no hope..