CS420+500: Advanced Algorithm Design and Analysis

Lectures: January 30, 2017

Prof. Will Evans *Scribe: Melody Wong*

In this lecture we went over:

- Residual Network
- Ford-Fulkerson Algorithm Proof

Handouts (posted on webpage):

- None given today

Suggested Reading:

- For more information on linear programming
  - chapter 7 of "Algorithms" by Dasguta, Papadimitriou and Vazirani
- For more information on network flow
  - See Erickson's Notes

Miscellaneous:

- "A Computer Scientist Thinks About the Brain" talk given by Papadimitriou.
  - Thursday, February 16, 2017 at 3:30 - 5:00 pm.
  - Life Sciences Building, Rm 1002.

1 Residual Network

- $G = (V, E)$ is the original flow network. V and E refer to the vertex and edge sets, respectively.
- $G^f = (V, E^f)$ is the flow network at flow f
  - Note that the edge set E^f is dependent on the flow, f

$$E^f = \{(u,v) : (u,v) \in E, f(u,v) < c(u,v) \text{ with capacity } c^f(u,v) = c(u,v) - f(u,v)\}$$
$$\bigcup \{(u,v) : (v,u) \in E, f(v,u) > 0 \text{ with capacity } c^f(u,v) = f(v,u)\}$$

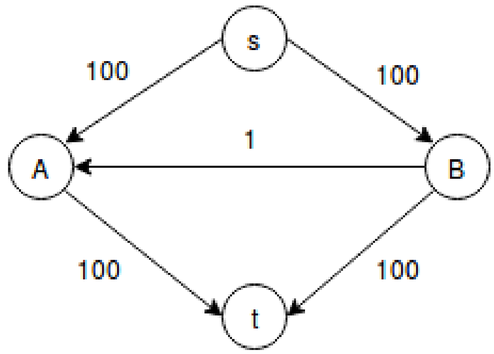


Figure 1: Flow network $G = (V, E)$

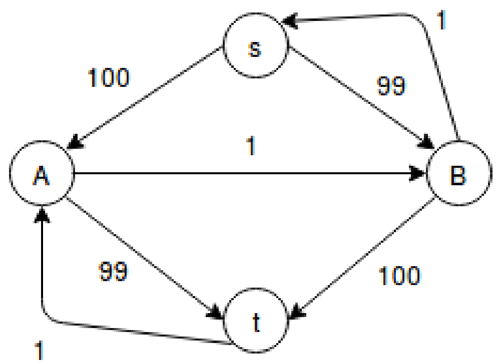


Figure 2: Residual network $G^f = (V, E^f)$. G^f is the residual network of G that has 1 unit flow from $s \to B \to A \to t$

Student Question Interlude

Q: How does Ford-Fulkerson find paths?
A: BFS! (Breadth first search)

Q: When does Ford-Fulkerson stop running?
A: It stops when you can no longer find a path from s to t

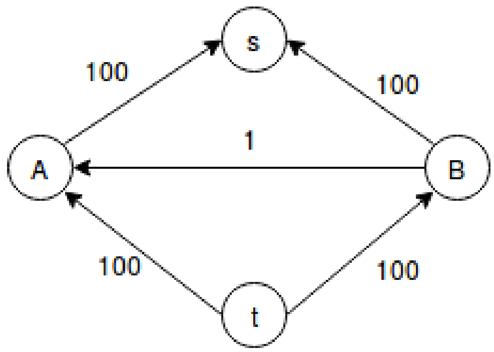
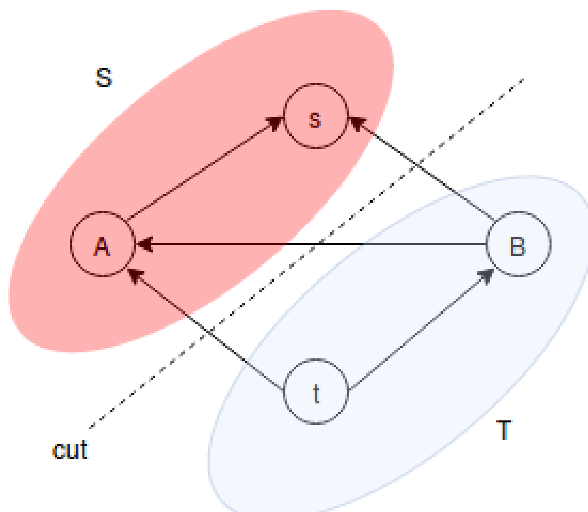


Figure 3: How residual graph looks like when you can no longer find a path from s to t

2 Ford-Fulkerson Algorithm Proof

- Definition: A **cut** is a partition (S, T) of V such that $s \in S, t \in T$
  - Note: We can partition in any way we want. We only need the souce in S and the sink in T.



- The **capacity** of the cut (S, T) is

$$cap(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)$$

LEMMA For any flow f and any cut (S, T)

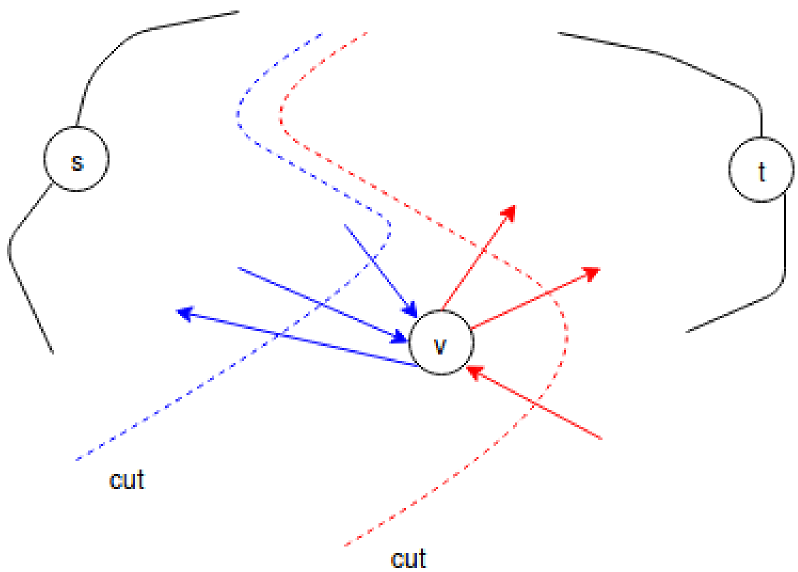
$$size(f) \equiv \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) \leq cap(S, T)$$

PROOF Any flow from s to t must cross from S to T and can't exceed $cap(S,T)$

Flow across cut S, T:

$$\begin{aligned}\sum_{u\in S}\sum_{v\in T} f(u,v) - f(v,u) &\leq \sum_{u\in S}\sum_{v\in T} f(u,v)\\ &\leq \sum_{u\in S}\sum_{v\in T} c(u,v)\end{aligned}$$

Move S,T cut while maintaining flow across cut until $S = \{s\}, T = V - \{s\}$. At that point, flow across cut equals $size(f)$.



Note: Flow conservation implies flow across red cut = flow across blue cut.