CS420+500: Advanced Algorithm Design and Analysis

Lectures: January 23 + 25, 2017

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In this lecture we discussed:

- Linear Programming;
- Simplex Method;
- Bandwidth Allocation Example.

1 Linear Programming

Motivating Example: Chocoalate Manufacturing
Consider a chocolate factory that can produce two types of boxed chocolate.
Box 1 has $1 profit, and demand is ≤ 200 boxes.
Box 2 has $6 profit, and demand is ≤ 300 boxes.
The factory is limited in production and cannot make more than 400 in one day.
How many of each box should be make to maximize profit?
Let $x_1 = \#$ boxes of box 1, and $x_2 = \#$ boxes of box 2.
We can then set up the following linear program:

$$\begin{aligned} \max\ x_1 &+ 6x_2 \\ x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned}$$

This can also be represented in matrix form, with $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$.

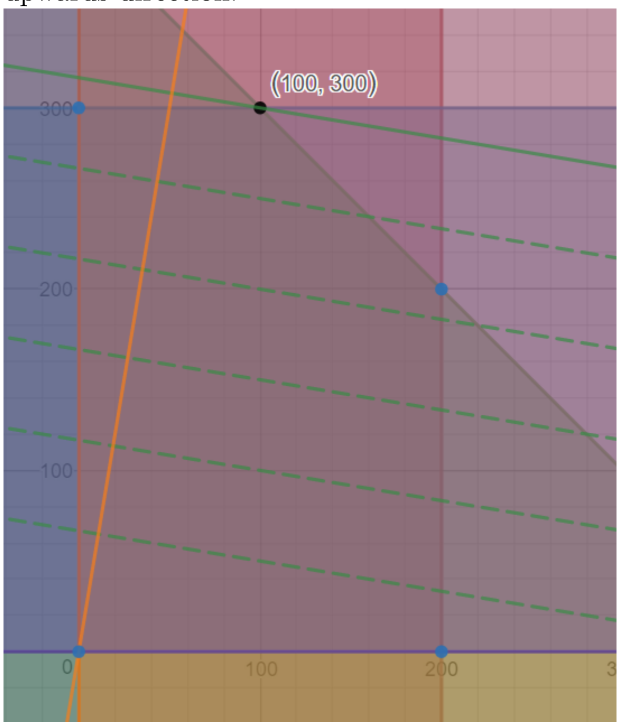
The general problem takes the form of

$$\max\ \boldsymbol{c} \cdot \boldsymbol{x}$$
$$\text{such that } \boldsymbol{Ax} \leq \boldsymbol{b}$$

Here, $x_1 + 6x_2$ is the objective function to be maximized, and the inequalities are constraints the objective function is restricted by.

Linear programming is type of problem where given a set of variables and constraints, how do we assign values to the variables such that:
1) They satisfy some given <u>linear</u> (in)equalities
2) They maximize a given <u>linear</u> objective function

When the inequalities are plotted (on the 2D plane in this case), we can obtain the feasible region, a convex polygon formed by the intersection of half planes. Below is the example for the motivating example.
Here, the vertices of the feasible region (or set) are marked by the blue and black points, the black being the optimal solution. The green lines are the constant profit curves, with the solid one being the curve the optimal solution falls on. The orange line is the direction of optimization, in the upwards direction.

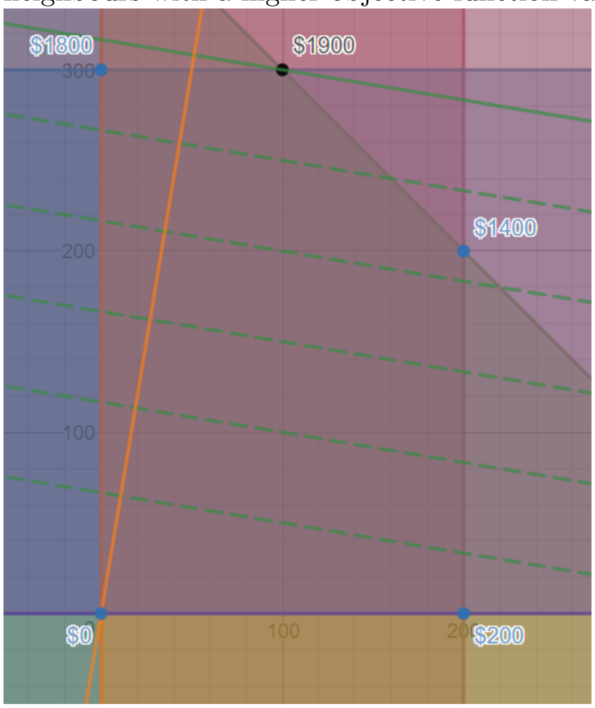


We get the constant profit curves by considering the equation $x_1 + 6x_2 = c$ for different values of c, which are represented by the different lines here. The optimal solution gives us the largest value of c, which is 1900 in this case.

2 The Simplex Method (Dantzig 1947)

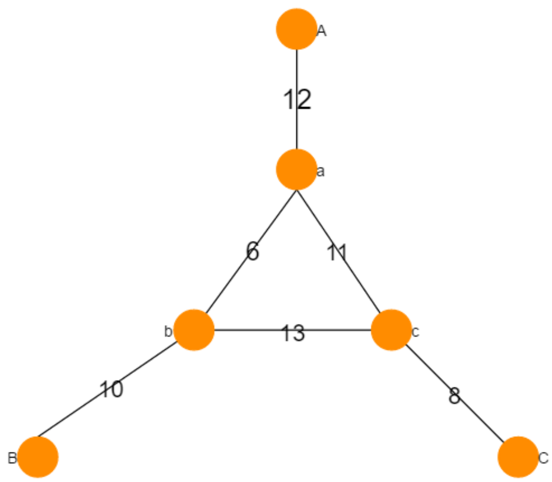
This is the algoritm generally used to solve linear programs, which goes as follows:
Start at a vertex v in the feasible set
While there is a neighbour v' of v such that v' has a higher objective function value, set $v = v'$
Return v

Below is the feasible region for the chocolate factory problem.
If we start at 0 and move in either direction, the algorithm will terminate at $1900 since it has no neighbours with a higher objective function value.



3 Bandwidth Allocation Problem

In this problem, we have to connect users A, B and C. Each pair needs at least 2 units between them, and there is a profit of \$3 for connecting A and B, \$2 for connecting B and C, and \$4 for connecting A and C.



Two paths satisfy each connection. For example, from A to B, there is a long path A $\rightarrow$ a $\rightarrow$ c $\rightarrow$ b $\rightarrow$ B, and a short path A $\rightarrow$ a $\rightarrow$ b $\rightarrow$ B. We will denote the bandwidth between A and B through the long path x'_{AB} and through the short path x_{AB}. We can then formulate the following LP:

$$\begin{aligned}
&\max x_{AB} + x'_{AB} + x_{BC} + x'_{BC} + x_{AC} + x'_{AC} \\
&\text{(Edge aA) } x_{AB} + x'_{AB} + x_{AC} + x'_{AC} \leq 12 \\
&\text{(Edge bB) } x_{AB} + x'_{AB} + x_{BC} + x'_{BC} \leq 10 \\
&\text{(Edge cC) } x_{AC} + x'_{AC} + x_{BC} + x'_{BC} \leq 8 \\
&\text{(Edge AB) } x_{AB} + x'_{BC} + x'_{AC} \leq 6 \\
&\text{(Edge BC) } x_{BC} + x'_{AC} + x'_{AB} \leq 13 \\
&\text{(Edge AC) } x_{AC} + x'_{BC} + x'_{AB} \leq 11 \\
&\text{At least 2 units} \quad x_{AB} + x'_{AB} \geq 2 \\
&x_{BC} + x'_{BC} \geq 2 \\
&x_{AC} + x'_{AC} \geq 2 \\
&\mathbf{x} \geq 0
\end{aligned}$$

The solution for this LP is $x_{AB} = 0, x'_{ABr} = 7, x_{BC} = x'_{BC} = 1.5, x_{AC} = 0.5, x'_{AC} = 4.5$.