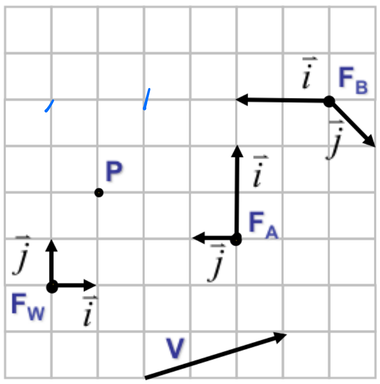
This midterm has 5 questions, for a total of 46 points.

1. Coordinate Frames



$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_W = \begin{bmatrix} 0 & -1 & 4 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_A$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_B = \begin{bmatrix} -1 & 0.5 & 2.5 \\ -2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_A$$

(a) (3 points) Express point P in each of the three coordinate frames.

$P_W(1,2)$ $P_A(0.5,3)$ $P_B(3.5,2)$

(b) (3 points) Express vector V in each of the three coordinate frames.

$V_W(3,1)$ $V_A(0.5,-3)$ $V_B(-2,-1)$

(c) (2 points) Find the 3×3 affine transformation matrix which takes a point from F_A and expresses it in terms of F_W. I.e., determine M, where $P_W = MP_A$. Write your answer in the space to the right of the diagram above.

(d) (2 points) Find the 3×3 affine transformation matrix which takes a point from F_A and expresses it in terms of F_B. I.e., determine M, where $P_B = MP_A$. Write your answer in the space to the right of the diagram above.

(e) (2 points) Develop a sequence of rotations, translates, and scales to construct the same 3×3 affine transformation as in part (c), i.e., $P_W = MP_A$. Express your solution as an algebraic composition of transformation matrices, in whatever order your prefer, e.g., $P_W = \text{Trans}(-1, 2, 0)\text{Rot}(z, 45°)\text{Scale}(2, 3, 1)P_A$.

Trans(4,1,0) Rot(Z, 90°) Scale(2,1,1)

2. Composition of Transformations

Consider the house which is shown below in its untransformed state, e.g., $F_h = F_W$, and the following transformations matrices:

$M_1 = \mathrm{Rot}(z, -90°)\mathrm{Scale}(2, 1, 1)\mathrm{Trans}(-1, 3, 0)$,

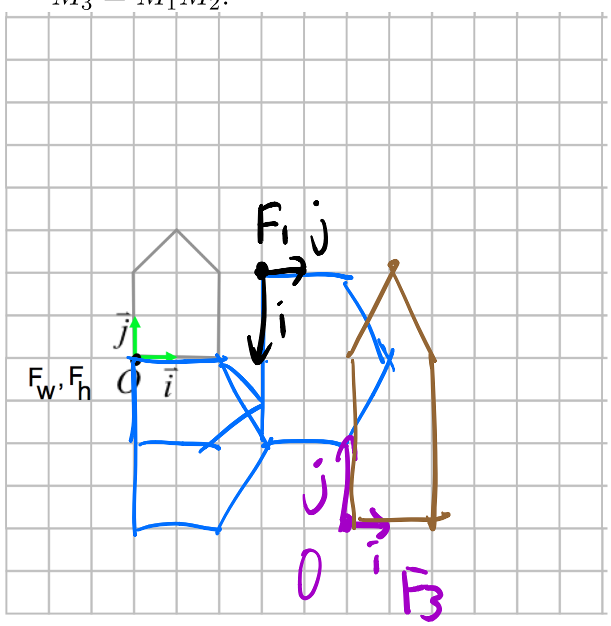
$$M_2 = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

Origin

j

i

$M_3 = M_1 M_2$.



(a) (2 points) Sketch the intermediate and final transformations of the house for the transformations that make up M_1. Draw the final coordinate frame as well, and label it with F_1.

(b) (2 points) Give an algebraic expression for M_1^{-1}, i.e., a transformation that would take a point from F_W to F_1.

Trans(1, -3, 0) Scale(0.5, 1, 1) Rot(z, 90°)

(c) (2 points) Sketch the coordinate frame and house that would result from applying transformation matrix M_3 to the house object, i.e., apply the M_2 transformation to your result in part (a). Label it with F_3.

(d) (3 points) Give 4×4 transformation matrices that perform the following:

Trans(3, 2, 5) Scale(9, 3, 4) Rot(z, 90)

$$\begin{bmatrix} 1 & & & 3 \\ & 1 & & 2 \\ & & 1 & 5 \\ & & & 1 \end{bmatrix} \quad \begin{bmatrix} 9 & & & \\ & 3 & & \\ & & 4 & \\ & & & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

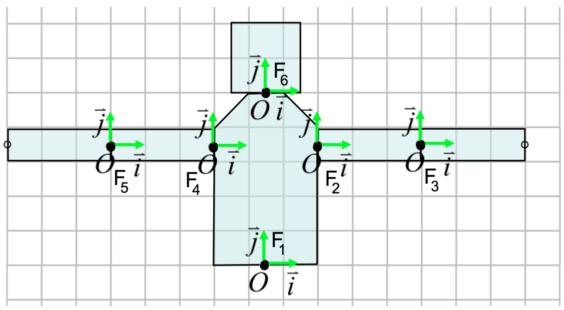
k

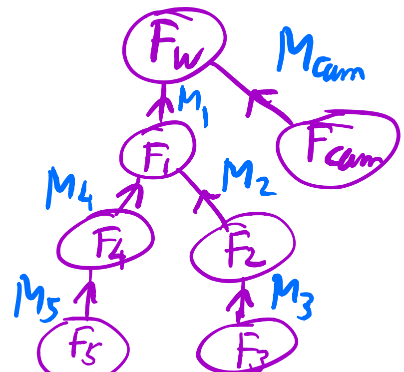
i

j

3. Scene Graphs

A 6-link human model is shown below. The links have the dimensions shown in the diagram. All of the links rotate about the Z-axis of their local coordinate frame by a specified angle θ_i. In the diagram $\theta_i = 0$ for all joints.





(a) (2 points) In the space above to the right, sketch a scene graph for the given scene. Place the world frame, F_W, at the root of your scene graph. Add a camera and treat it as any other object. Label the frames as F_W, F_1, F_2, F_3, F_4, F_5, F_6, and F_{cam}. Add labels to the edges, using M_i to designate the transformation matrix that takes a point in frame i to its parent frame. Use arrows to indicate the direction of the change-of-basis transformation.

(b) (2 points) Give an algebraic expression in terms of the matrices M_i for the compound transformation, $M_{5\to\text{cam}}$ that transforms a point from frame F_5 to camera coordinates, F_{cam}. Then give an expression for the opposite transformation, $M_{\text{cam}\to 5}$.

$$P_{cam} = M_{cam}^{-1} M_1 M_4 M_5 P_5$$
$$P_5 = M_5^{-1} M_4^{-1} M_1^{-1} M_{cam} P_{cam}$$

(c) (2 points) Suppose we wanted to determine the distance, d, between the two hands (shown as circles), for some pose. Describe how to compute d. Assume that all the transformation matrices, M_i, are known.

$P_{RH} = (3,0,0)$ $P_{LH} = (-3,0,0)$

in F_5 $\longrightarrow$ $P'_{RH} = M_5^{-1} M_4^{-1} M_2 M_3 P_{RH}$ Then $d = ||P'_{RH} - P_{LH}||$

(d) (1 point) Give the sequence of translates and rotates needed to build matrix M_3, i.e., that connects link 3 to link 2. Take into account θ_3 as well.

$$M_3 = Trans(3,0,0)\, Rot(z, \theta_3)$$

4. (5 points) True/False and short answer

   T/F (true or false):

   (a) **T** With a pinhole camera model, objects are always in focus.

   (b) **F** In a right-handed coordinate system, $k \times j = i$.

   (c) **F** The matrix $M = 2I$, where I is a 4×4 identity marix, produces a transformation that is equivalent to `Scale(2,2,2)`.

   False because h also gets scaled.

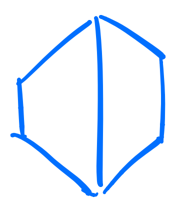
   Short answer:

   (d) Suppose that we have a set of points $\{P_i\}$ to which we apply an affine transformation M to produce a set of transformed points, $\{P_i'\}$. Given only these corresponding points before-and-after the transformation, how many points are needed to uniquely determine the affine transformation M? **4** $M \in \mathbb{R}^{12}$ therefore we need 12 eqns: 4 3D pts.

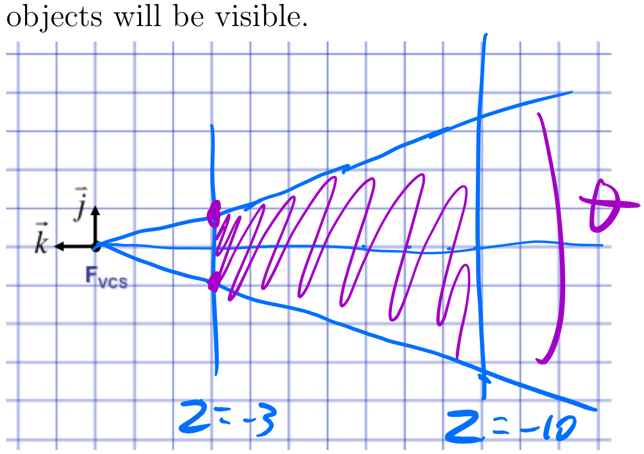
   (e) Express the point $(x, y, z, h) = (-6, 2, 3, 3)$ in cartesian coordinates. $(-2, \frac{2}{3}, 1)$

5. Projection Transformations

   (a) (2 points) Sketch a two-point perspective projection of a cube.



   (b) (3 points) Sketch a side view, i.e., yz-plane, of a perspective view volume defined by `near=3, far=10, top=1, bot=-1, left=-1, right=1`. Shade the region where objects will be visible.



(c) (1 point) What is the vertical field of view, given as an angle, of the above view frustum? Giving your answer in terms of an `arctan()` is fine.

$\theta = 2\,\mathrm{atan}(1/3)$

(d) (4 points) The point $P(1, 0, 3)$, given in local object coordinates, is transformed using the modelview and projection matrices given below, and onto a 1000×1000 display window. What is the final display location of this point? Show your work, i.e., give all the intermediate results.

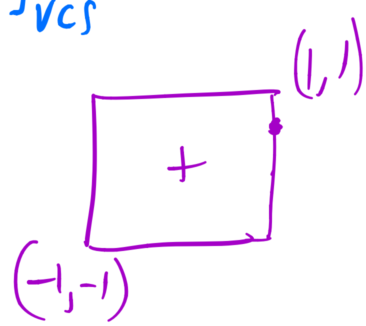
$$M_{modelview} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, M_{proj} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5/3 & -8/3 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 3 \\ 2 \\ -3 \\ 1 \end{bmatrix}_{VCS} = \begin{bmatrix} 2 & & & 1 \\ & 1 & & 2 \\ & & 1 & -6 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}_{OCS}$$

$$\begin{bmatrix} 3 \\ 2 \\ 7/3 \\ 3 \end{bmatrix}_{CCS} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -5/3 & -8/3 \\ & & -1 & \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -3 \\ 1 \end{bmatrix}_{VCS}$$

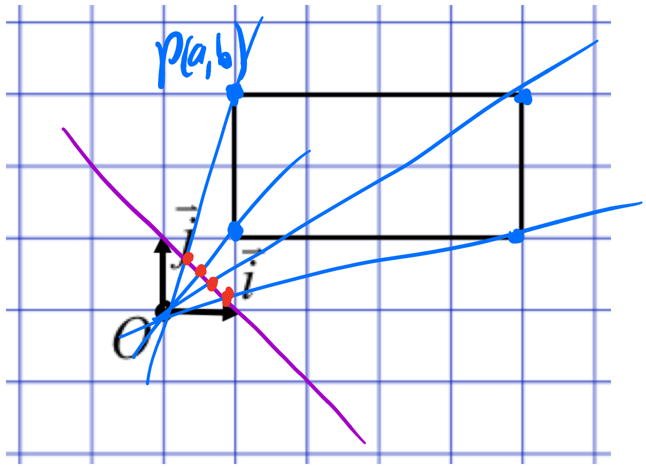
$$\xrightarrow{/h} \begin{bmatrix} 1 \\ 2/3 \\ 7/9 \end{bmatrix}_{NDCS}$$

$$\begin{bmatrix} 1000 \\ (5/6)\,1000 \end{bmatrix}_{DCS}$$



(e) (3 points) Suppose that we wish to project points $P(x, y)$ onto the line $y = 1 - x$. As usual, assume that the origin is the center of projection.

First, use the diagram below to graphically illustrate the result of projecting the four corner points of the object onto the desired line. Second, develop a 3×3 projection matrix (for our 2D case) that implements this projection.



line: $y' = 1 - x'$

projector: $y' = \left(\frac{b}{a}\right) x'$

$\Rightarrow 1 - x' = \left(\frac{b}{a}\right) x'$

$1 = x'\left(1 + \frac{b}{a}\right)$

$1 = x'\left(\frac{a+b}{a}\right)$

$x' = \frac{a}{a+b}$

$y' = \frac{b}{a+b}$

$$\begin{bmatrix} a \\ b \\ a+b \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$$

$$\xrightarrow{/h} \begin{bmatrix} a/(a+b) \\ b/(a+b) \end{bmatrix}$$