CPSC 314
Assignment 2

due: Wednesday, October 3, 2018, 11:59pm

Answer the questions in the spaces provided on the question sheets.

Name: Solutions

Student Number:

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Question 2	/ 8
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TOTAL	/ 50

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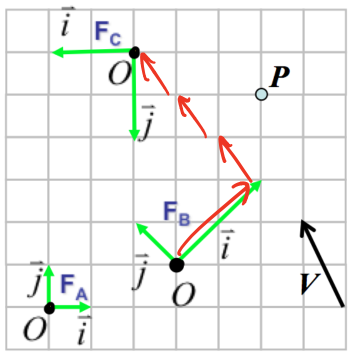
$\frac{2j-i}{2}$ $\qquad$ $\frac{2j+i}{-2} = -j - \frac{i}{2}$

grid $\qquad$ i_B, j_B

1. Transformations as a change of coordinate frame



p

$$\begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}_A = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \end{bmatrix}_C$$

$$\begin{bmatrix} 3/2 \\ 1 \\ 1 \end{bmatrix}_B = \begin{bmatrix} -1/2 & -1/2 & 1 \\ 1 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \end{bmatrix}_C$$

$$V_B: \; i_w = \frac{i-2j}{4}$$
$$j_w = \frac{i+2j}{4}$$
$$V_B = -i_w + 2j_w = \frac{(2j-i) + 2(2j+i)}{4} = \frac{i}{4} + \frac{6j}{4} = \left(-\frac{3}{4}, \frac{3}{2}\right)$$

(a) (3 points) Express the coordinates of point P with respect to coordinate frames A, B, and C.

$P_A(5,5)$ $\qquad$ $P_B(1.5, 1)$ $\qquad$ $P_C(-1.5, 0.5)$

(b) (3 points) Express the coordinates of vector V with respect to coordinate frames A, B, and C.

$V_A(-1,2)$ $\qquad$ $V_B(\frac{1}{4}, \frac{3}{2})$ $\qquad$ $V_C(0.5, -1)$

(c) (3 points) Fill in the 2D transformation matrix, $M_{C\to A}$, that takes points from F_C to F_A, as given to the right of the above figure.

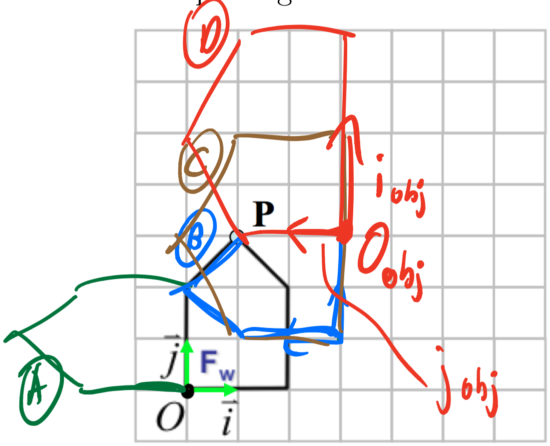
(d) (3 points) Fill in the 2D transformation matrix, $M_{C\to B}$, that takes points from F_C to F_B, as given to the right of the above figure.

(e) (2 points) Given an expression for the 2D transformation matrix, $M_{A\to B}$, in terms of the matrices $M_{C\to A}$ and $M_{C\to B}$. Then evaluate the expression using your results above. Use any tool you like to compute the required matrix inverse, i.e., matlab, an online web page, or you can also develop the matrix yourself in the same way you developed the other matrices. Test your solution using point P.

$$M_{A\to B} = M'_{C\to B} M_{A\to C}$$
$$= \begin{bmatrix} -1/2 & -1/2 & 1 \\ 1 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & & 2 \\ & -2 & 6 \\ & & 1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} -1/2 & -1/2 & 1 \\ 1 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & & 1 \\ & -1/2 & 3 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & -1 \\ -1/2 & 1/2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3/2 \\ 1 \\ 1 \end{bmatrix}_B = \begin{bmatrix} 1/4 & 1/4 & -1 \\ -1/2 & 1/2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}_A$$

2. Interpreting a Matrix Transformation



$$\begin{array}{c} \\ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_w \end{array} = \begin{array}{c} \begin{array}{ccc} i_{obj} & j_{obj} & O_{obj} \end{array} \\ \begin{bmatrix} 0 & -1 & 3 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{obj} \end{array}$$

(a) (2 points) On the above diagram, sketch the origin and basis vectors of the coordinate frame F_{obj} that results from the following sequence of transformations that develops a transformation matrix $P_W = MP_{obj}$. Assume that all transformations applied to M do right-multiplication.

```
  M.identity();
A M.rotate(Z,90);
B M.translate(1,-3,0);
C M.scale(2,1,1);
D M.translate(1,0,0);
```

(b) (2 points) The drawing of the above house represents its untransformed shape. Sketch the transformed version of the house in the above diagram.

(c) (2 points) Give the values of the resulting transformation matrix, M. Complete this in the space given to the right of the diagram.

(d) (2 points) Suppose that an alternate sequence of transformations given by
$M = Translate(a, b, 0)Rotate(Z, \theta)Scale(c, d, 1)$,
is used to implement the same transformation. Provide the values of a, b, c, d and θ that would implement the given transformation.

$M = Translate(3,3,0)\ Rotate(z, 90^\circ)\ Scale(2,1,0)$

$a = 3$ $\quad \theta = 90^\circ$
$b = 3$
$c = 2$
$d = 1$

3. Affine transformations

(a) (2 points) Develop a 3D affine transformation that scales around the point (x, y, z) by a factor of (s_x, s_y, s_z). Express your solution as a sequence of transformations.

$$Trans(x,y,z)\ Scale(S_x, S_y, S_z)\ Trans(-x,-y,-z)$$

(b) (2 points) Suppose that a scene has 1,000,000 vertices to be rendered at 60 Hz. The vertex shader applies two successive 4×4 transformations to each vertex. How many multiplications per second does the vertex shader need to perform? Assume no special optimizations. Show your work.

$$p' = (M_1(M_2 p)) \rightarrow 2\ M \cdot V \text{ multiplies} = 2 \times 16 \text{ scalar multiplies} = \boxed{32}$$

$$p' = ((M_1 M_2)\, p) \rightarrow M\text{-}M \text{ multiply} + M \cdot V = 64 + 16 = \boxed{80}$$

$$\times 60 \text{ million} = \boxed{4.8 \cdot 10^9} \text{ or } \boxed{1.92 \times 10^9} \text{ mult/sec}$$

(c) (2 points) In 2D, show that a matrix $M = Rotate(Z, \theta)Scale(a, b)$ has orthogonal basis vectors, whereas the same is not true for the reversed order, i.e., $M = Scale(a, b)Rotate(Z, \theta)$.

$$R \cdot S = \begin{bmatrix} c & -s & \\ s & c & \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & & \\ & b & \\ & & 1 \end{bmatrix} = \begin{bmatrix} ac & -bs & 0 \\ as & bc & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad i \cdot j = -abcs + abcs = 0$$

$$c = \cos(\theta) \qquad s = \sin(\theta)$$

$$S \cdot R = \begin{bmatrix} a & & \\ & b & \\ & & 1 \end{bmatrix} \begin{bmatrix} c & -s & \\ s & c & \\ & & 1 \end{bmatrix} = \begin{bmatrix} ac & -as & 0 \\ bs & bc & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad i \cdot j = -a^2cs + b^2cs = cs(b^2 - a^2) \neq 0$$

(d) (2 points) Can the transformation that is implemented by $M = Translate(a, b, c)Rotate(X, \theta_1)Rotate(Y, \theta_2)Rotate(Z, \theta_3)Scale(d, e, f)$ be used to implement an arbitrary 3D affine transformation? Why or why not? If not, give an example of a 3D affine transformation that could not be implemented using the above sequence of transformations.

Argument ①: An affine transformation is defined by 12 numbers:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{12}$$

The above expression only has 9 free parameters $(a, b, c, d, e, f, \theta_1, \theta_2, \theta_3)$

Argument ②: The above transformations leave the local basis vectors orthogonal to each other.

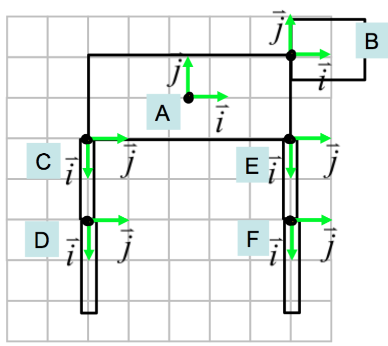
Example: Suppose we want $i = j = k = \langle 1, 0, 0 \rangle$, Origin $= (0,0,0)$

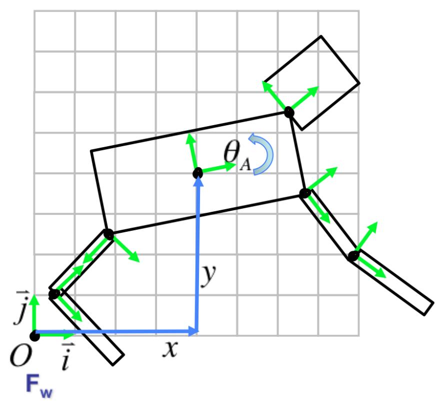
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

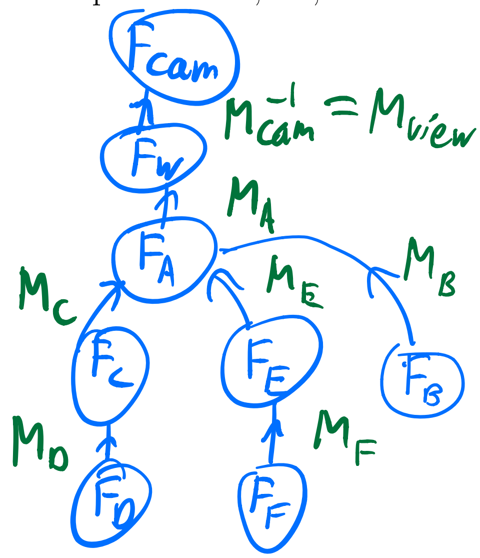
The above transformations cannot map j and k onto i.

4. Scene Graphs

(a) (3 points) In the space to the right below, sketch a scene graph for the simple 2D dog. The labeled nodes should represent the coordinate frames, e.g., F_A, for each link. Use directed edges, i.e., arrows, to represent transformations that connect the nodes in the graph. Label each edge with a unique name, e.g., M_A, which indicates the transformation matrix that takes points from the given frame to its parent frame. Use the world coordinate frame as the root note of the scene graph. Assume that the body, F_A, and the camera, F_{cam}, are positioned relative to the world frame, F_{world}. Assume that all other parts are defined relative to their parent links, i.e., the links closer to the body.







(b) (2 points) Give an algebraic expression for the composite transformation that would be used when drawing a point P_D, as defined in frame F_D.i.e., it should transform point P_D to the camera coordinate frame, F_{cam}. Your answer should be expressed as a product of the matrices used to label your scene graph.

$$P_{cam} = M_{cam}^{-1} M_A M_C M_D P_D$$

(c) (2 points) Similarly, give a algebraic expression for the composite transformation that takes point, P_D, as defined in frame F_D, to the head frame, F_B.

$$P_B = M_B^{-1} M_C M_D P_D$$

(d) (3 points) The full position of the dog is described by $x, y, \theta_A, \theta_B, \theta_C, \theta_D, \theta_E, \theta_F$, where all angles specify rotations about the Z-axis, relative to their parent link. The figure on the left shows the dog in its reference configuration, i.e., with $\theta_A, \theta_B, \theta_C, \theta_D, \theta_E, \theta_F = 0$.
Give expressions for each of M_A, M_B, M_C, M_D, M_E and M_F in terms of a product of translations and rotation matrices, written as `Translate(a,b,0)` and `Rotate(z,θ)`. Assume ~~that~~ no scaling.

$M_A = \text{Trans}(x,y,0)\, \text{Rot}(z,\theta_A)$

$M_B = \text{Trans}(2.5,1,0)\, \text{Rot}(z,\theta_B)$

$M_C = \text{Trans}(-2.5,-1,0)\, \text{Rot}(z,-90^\circ+\theta_C)$

$M_D = \text{Trans}(2,0,0)\, \text{Rot}(z,\theta_D)$

$M_E = \text{Trans}(2.5,-1,0)\, \text{Rot}(z,-90^\circ+\theta_E)$

$M_F = \text{Trans}(2,0,0)\, \text{Rot}(z,\theta_F)$

5. A requirement of a 3×3 rotation matrix, $R = \begin{bmatrix} \vec{\mathbf{a}} & \vec{\mathbf{b}} & \vec{\mathbf{c}} \end{bmatrix}$, is that the columns, $\mathbf{a}, \mathbf{b}, \mathbf{c}$, have unit magnitude and are mutually orthogonal, i.e., a zero dot product. Furthermore, for a right-handed coordinate system, we require $\mathbf{a} \times \mathbf{b} = \mathbf{c}$.

(a) (1 point) Given a rotation matrix defined by three column vectors, $R = \begin{bmatrix} \vec{\mathbf{a}} & \vec{\mathbf{b}} & \vec{\mathbf{c}} \end{bmatrix}$ compute the resulting matrix product, $M = R^T R$.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a\cdot a & a\cdot b & a\cdot c \\ b\cdot a & b\cdot b & b\cdot c \\ c\cdot a & c\cdot b & c\cdot c \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} = I$$

(b) (2 points) A 4×4 rigid body transformation is defined by a rotation matrix and a translation, T, as shown below. Develop an expression for the inverse of this transformation matrix. Hint: your answer to part (a) provides most of the solution.

$$\begin{bmatrix} d_x & e_x & f_x & T_x \\ d_y & e_y & f_y & T_y \\ d_z & e_z & f_z & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} & d & & -d\cdot T \\ & e & & -e\cdot T \\ & f & & -f\cdot T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(See the course notes on the viewing matrix)

(c) (3 points) Determine if the matrices below are rotations. Why or why not?

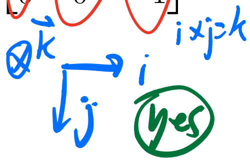
(i) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Check: $i \cdot j = j \cdot k = i \cdot k = 0$

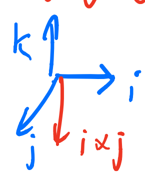
$\|i\| = \|j\| = \|k\| = 1$

$i \times j = k$

(i) $|i| \neq 1$ no

(ii) 

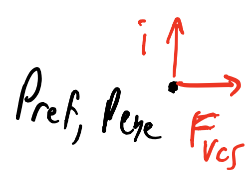
$i \times j = k$ yes

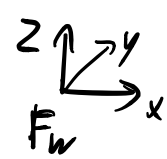
(iii) 

no $i \times j = -k$

6. (4 points) Viewing Transformation

Determine the viewing transformation, M_{view}, that takes points from WCS (world coordinates) to VCS (viewing or camera coordinates), for the following camera parameters: $P_{eye} = (5, 10, 10)$, $P_{ref} = (5, 0, 10)$, $V_{up} = (1, 0, 0)$. Show your work.





$K = P_{eye} - P_{ref} = \langle 0, 10, 0 \rangle$

$\vec{k} = K / \|K\| = \langle 0, 1, 0 \rangle$

$\vec{I} = \vec{V_{up}} \times \vec{k} = \langle 1,0,0 \rangle \times \langle 0,1,0 \rangle = \langle 0,0,1 \rangle = \vec{i}$

$\vec{j} = \vec{k} \times \vec{i} = \langle 0,1,0 \rangle \times \langle 0,0,1 \rangle = \langle 1,0,0 \rangle$

$$M_{view} = \left(\begin{bmatrix} 1 & & & 5 \\ & 1 & & 10 \\ & & 1 & 10 \\ & & & 1 \end{bmatrix} \begin{array}{c} \begin{array}{cccc} i & j & k & \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \right)^{-1} = \begin{bmatrix} 0 & 0 & 1 & -10 \\ 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

See Q5(b)