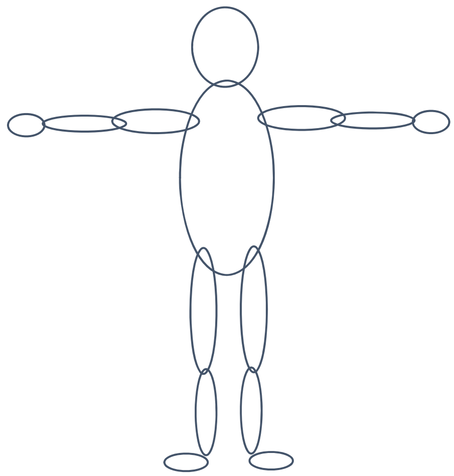


CPSC 314
09 – COMPOSITE TRANSFORMATIONS. HIERARCHIES

TEXTBOOK: 1.4, 5.1

UGRAD.CS.UBC.CA/~CS314

Alla Sheffer
2016

THEORY &PROGRAMMING A2

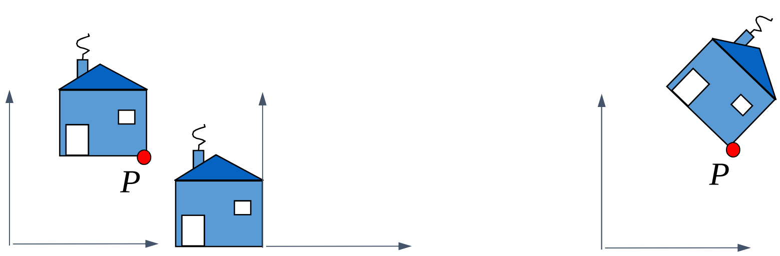
- Out
- Theory
  - Matrices!
  - Due end of the week (Fri, Oct 7^{th})
  - Best preparation for the midterm
- Programming
  - Animate a simple octopus
  - More of transformation matrices, yum!

TRANSFORMATION COMPOSITION

- What operation rotates XY by θ around $P = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$?

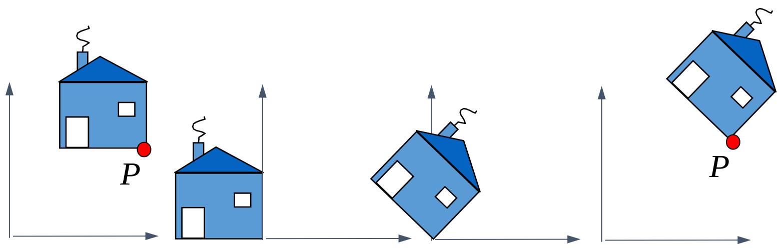
TRANSFORMATION COMPOSITION

- What operation rotates XY by θ around $P = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$?
- Answer:
  - Translate P to origin



TRANSFORMATION COMPOSITION

- What operation rotates XY by $\theta < 0$ around $P = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$?
- Answer:
  - Translate P to origin
  - Rotate around origin by θ
  - Translate back



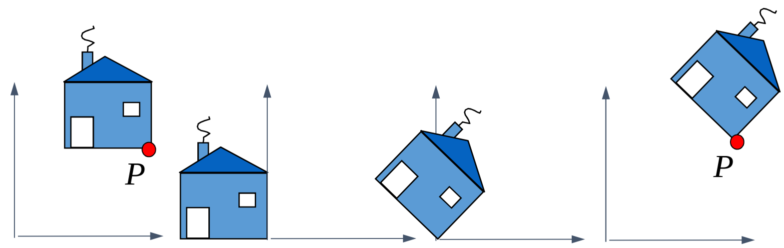
TRANSFORMATION COMPOSITION

$$T^{(p_x, p_y)} R^{\theta} T^{(-p_x, -p_y)}(V)$$

$$= \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$

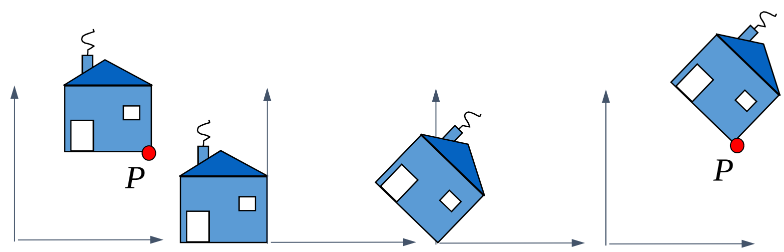
TWO INTERPRETATIONS

$$\begin{pmatrix} \cos\theta & -\sin\theta & p_x \cdot (1-\cos\theta) + p_y \cdot \sin\theta \\ \sin\theta & \cos\theta & p_y \cdot (1-\cos\theta) + p_x \cdot \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$



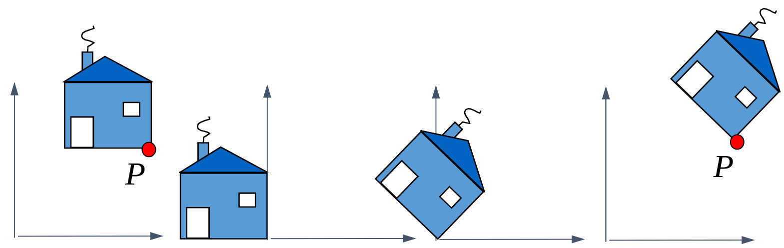
TRANSFORMING COORDINATES

$$\begin{pmatrix} \cos\theta & -\sin\theta & p_x \cdot (1-\cos\theta) + p_y \cdot \sin\theta \\ \sin\theta & \cos\theta & p_y \cdot (1-\cos\theta) + p_x \cdot \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$



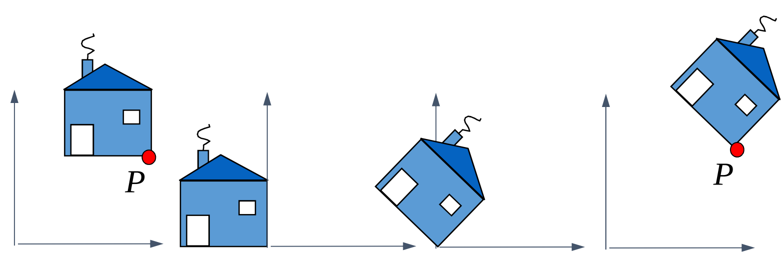
TRANSFORMING COORDINATES

$$\begin{pmatrix} \cos\theta & -\sin\theta & p_x\cdot(1-\cos\theta)+p_y\cdot\sin\theta \\ \sin\theta & \cos\theta & p_y\cdot(1-\cos\theta)+p_x\cdot\sin\theta \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$



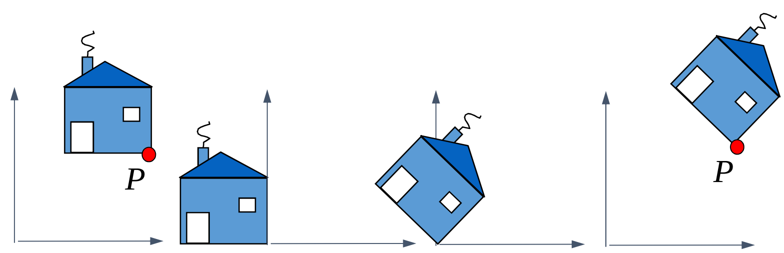
TRANSFORMING COORDINATES

$$\begin{pmatrix} \cos\theta & -\sin\theta & p_x\cdot(1-\cos\theta)+p_y\cdot\sin\theta \\ \sin\theta & \cos\theta & p_y\cdot(1-\cos\theta)+p_x\cdot\sin\theta \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$



TRANSFORMING COORDINATE FRAME

$$\begin{pmatrix} \cos\theta & -\sin\theta & p_x \cdot (1-\cos\theta) + p_y \cdot \sin\theta \\ \sin\theta & \cos\theta & p_y \cdot (1-\cos\theta) + p_x \cdot \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$



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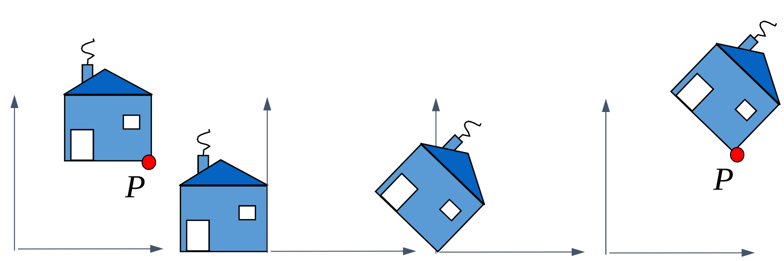

TRANSFORMING COORDINATE FRAME

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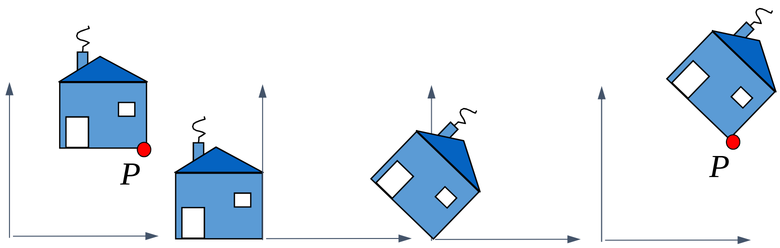

TRANSFORMING COORDINATE FRAME

Columns are new basis vectors (and new origin)!

$$\begin{pmatrix} \cos\theta & -\sin\theta & p_x \cdot (1-\cos\theta) + p_y \cdot \sin\theta \\ \sin\theta & \cos\theta & p_y \cdot (1-\cos\theta) + p_x \cdot \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$

TRANSFORMING COORDINATE FRAME

$$T^{(p_x,p_y)}R^{\theta}T^{(-p_x,-p_y)}\begin{pmatrix}v_x\\v_y\\1\end{pmatrix}$$

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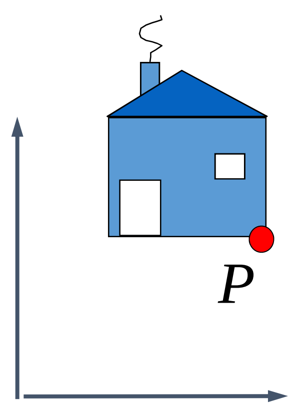
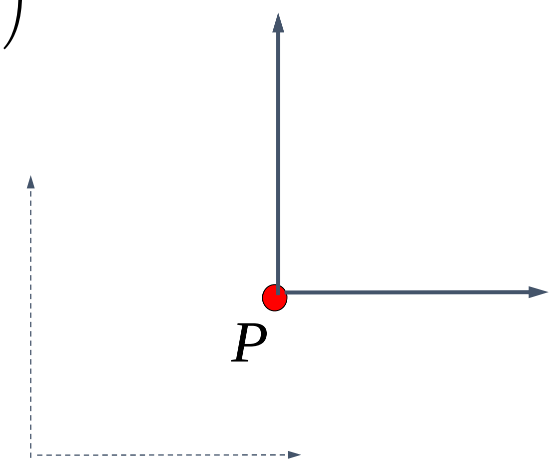


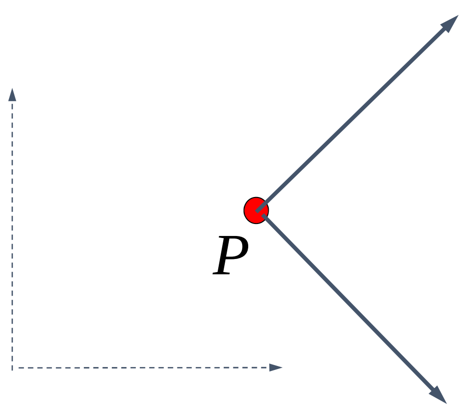
TRANSFORMING COORDINATE FRAME

$$\boldsymbol{T}^{(p_x,p_y)} R^{\theta} T^{(-p_x,-p_y)} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$



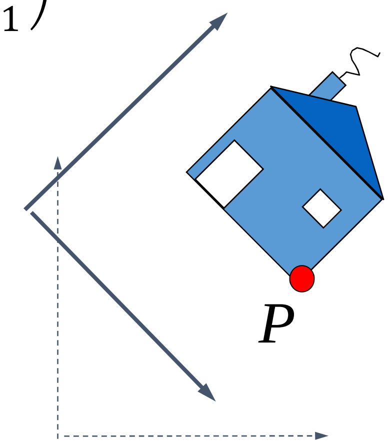
TRANSFORMING COORDINATE FRAME

$$T^{(p_x,p_y)} \boldsymbol{R}^{\boldsymbol{\theta}} T^{(-p_x,-p_y)} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$



TRANSFORMING COORDINATE FRAME

$T^{(p_x,p_y)} R^{\theta} \boldsymbol{T^{(-p_x,-p_y)}} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$

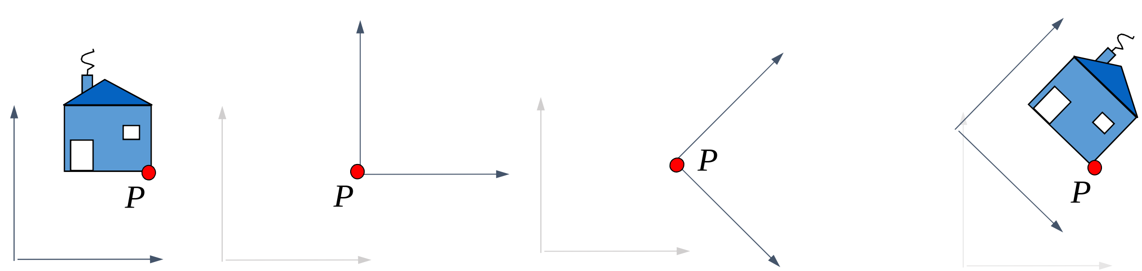


TRANSFORMING COORDINATE FRAME

World Coordinate Frame

Object Coordinate Frame

$$\begin{pmatrix} v'_x \\ v'_y \\ 1 \end{pmatrix} = T^{(p_x,p_y)} R^{\theta} T^{(-p_x,-p_y)} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$



TWO INTERPRETATIONS OF COMPOSITE

World Coordinate Frame — Object Coordinate Frame

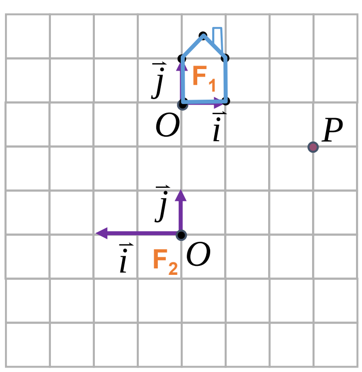
$$\begin{pmatrix} v'_x \\ v'_y \\ 1 \end{pmatrix} = T^{(p_x,p_y)} R^{\theta} T^{(-p_x,-p_y)} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$$

1) read from inside-out as transformation of object

2) read from outside-in as transformation of the coordinate frame

TRANSFORMATIONS - EXAMPLE

change of basis expressed using a matrix

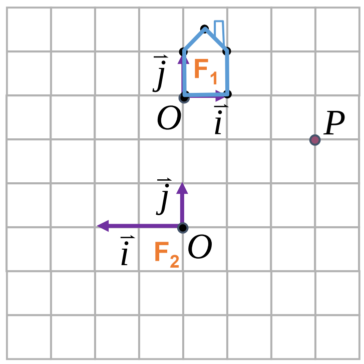


$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_1 + x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}_1 + y_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}_1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}_2 + x_1 \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}_2 + y_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}_2$$

TRANSFORMATIONS - EXAMPLE

change of basis expressed using a matrix



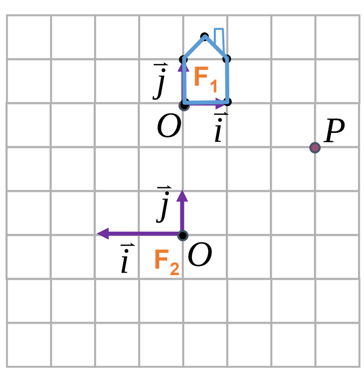
$$\begin{bmatrix}x_1\\y_1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}_1 + x_1\begin{bmatrix}1\\0\end{bmatrix}_1 + y_1\begin{bmatrix}0\\1\end{bmatrix}_1$$

$$\begin{bmatrix}x\\y\end{bmatrix}_2 = \begin{bmatrix}0\\3\end{bmatrix}_2 + x_1\begin{bmatrix}-0.5\\0\end{bmatrix}_2 + y_1\begin{bmatrix}0\\1\end{bmatrix}_2$$

$$\begin{pmatrix}x_2\\y_2\\1\end{pmatrix} = \begin{pmatrix}-0.5 & 0 & 0\\0 & 1 & 3\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\y_1\\1\end{pmatrix}$$

TRANSFORMATIONS - EXAMPLE

change of basis expressed using a matrix



$$\begin{bmatrix}x_1\\y_1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}_1 + x_1\begin{bmatrix}1\\0\end{bmatrix}_1 + y_1\begin{bmatrix}0\\1\end{bmatrix}_1$$

$$\begin{bmatrix}x\\y\end{bmatrix}_2 = \begin{bmatrix}0\\3\end{bmatrix}_2 + x_1\begin{bmatrix}-0.5\\0\end{bmatrix}_2 + y_1\begin{bmatrix}0\\1\end{bmatrix}_2$$

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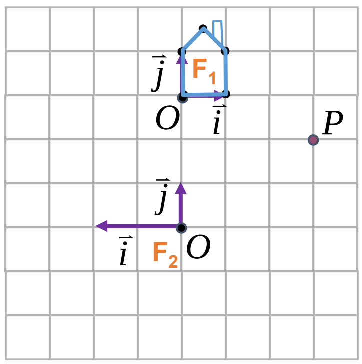
TRANSFORMATIONS - EXAMPLE

change of basis expressed using a matrix



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_1 + x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}_1 + y_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}_1$$

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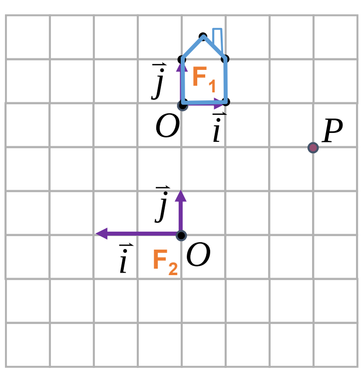
TRANSFORMATIONS - EXAMPLE

change of basis expressed using a matrix

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_1 + x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}_1 + y_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}_1$$



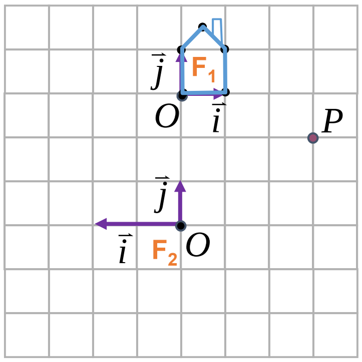
$$\begin{bmatrix} x \\ y \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}_2 + x_1 \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}_2 + y_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}_2$$

$\vec{i_1}$ $\vec{j_1}$ O_1

$$\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix}$$

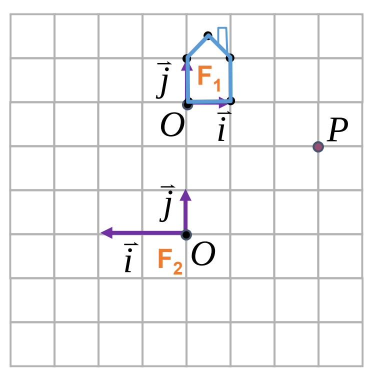
TRANSFORMATIONS - EXAMPLE

- How to transform F_2 into F_1?



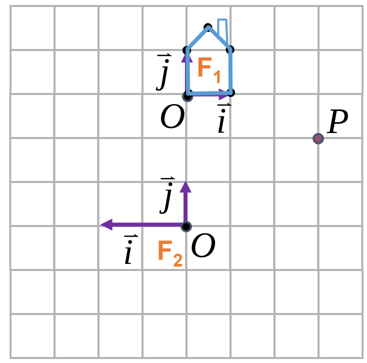
TRANSFORMATIONS - EXAMPLE

- How to transform F_2 into F_1?
  - Scale by (-0.5, 1)



TRANSFORMATIONS - EXAMPLE

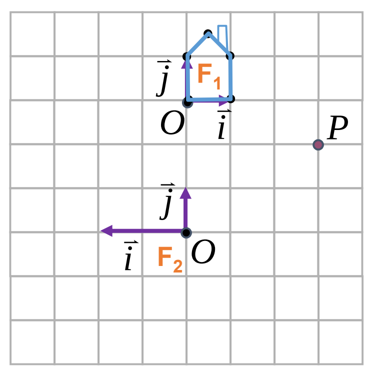
- How to transform F_2 into F_1?
  - Scale by (-0.5, 1)
  - Translate by (0,3)



TRANSFORMATIONS - EXAMPLE

- How to transform F_2 into F_1?
  - Scale by (-0.5, 1)
  - Translate by (0,3)

$$M = M_{scale} \cdot M_{translate}$$

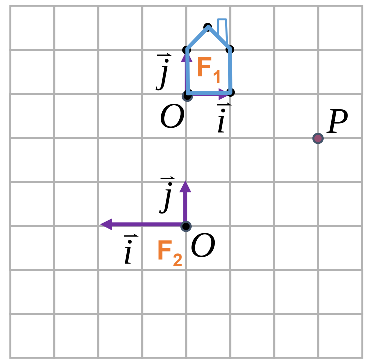


TRANSFORMATIONS - EXAMPLE

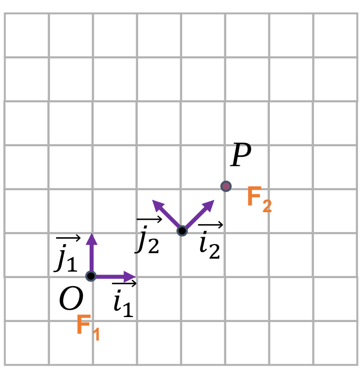
- How to transform F_2 into F_1?
  - Scale by (-0.5, 1)
  - Translate by (0,3)

$$M = M_{scale} \cdot M_{translate} = \begin{pmatrix} -0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} =$$

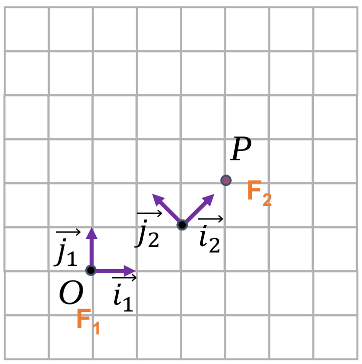
$$\begin{pmatrix} -0.5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$



ANOTHER EXAMPLE

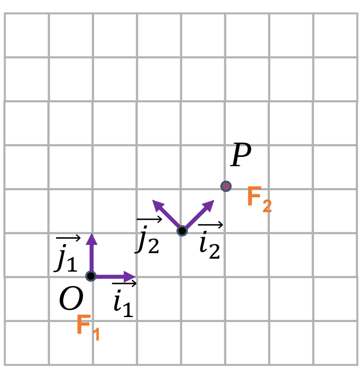


ANOTHER EXAMPLE



$$P = \begin{pmatrix} 3 \\ 2 \end{pmatrix}_1 = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}_2$$

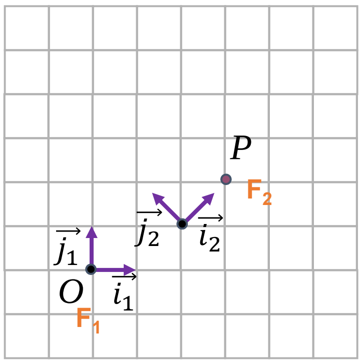
ANOTHER EXAMPLE



$$P = \begin{pmatrix} 3 \\ 2 \end{pmatrix}_1 = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}_2$$

$$M_{21} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

ANOTHER EXAMPLE



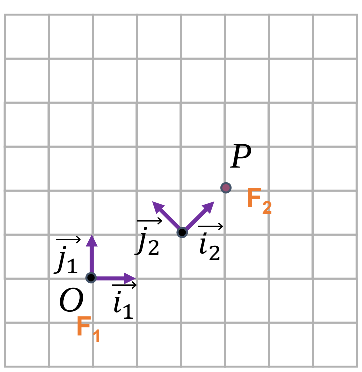
$$P = \begin{pmatrix} 3 \\ 2 \end{pmatrix}_1 = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}_2$$

Converts coordinates from 2 into 1

$$M_{21} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

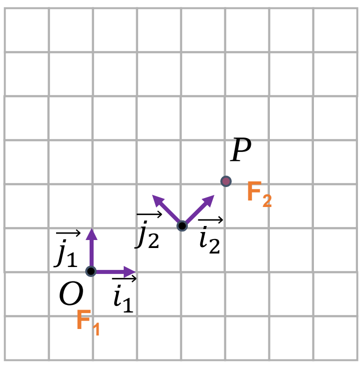
$$M_{21} \cdot \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

ANOTHER EXAMPLE



To convert point from 2 into 1, let's think the other way:
how to get to coordinate frame 2 from 1?

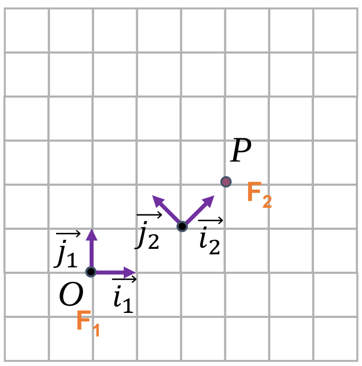
ANOTHER EXAMPLE



To convert point from 2 into 1, let's think the other way:
how to get to coordinate frame 2 from 1?

$$M_{21} = Tr_{(2,1)} \cdot Rot_{\pi/4}$$

ANOTHER EXAMPLE



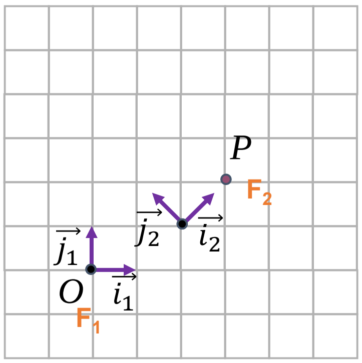
To convert point from 2 into 1, let's think the other way:
how to get to coordinate frame 2 from 1?

$$M_{21} = Tr_{(2,1)} \cdot Rot_{\pi/4} =$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) & 0 \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

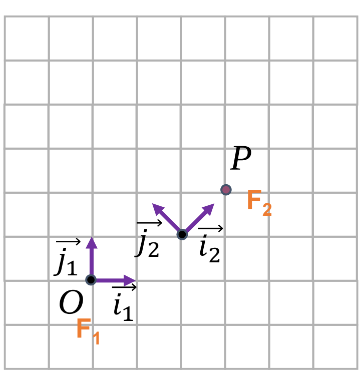
ANOTHER EXAMPLE



To convert point from 2 into 1, let's think the other way: how to get to coordinate frame 2 from 1?

$$M_{21} = Tr_{(2,1)} \cdot Rot_{\pi/4} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

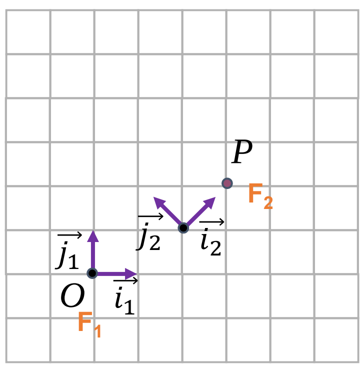
ANOTHER EXAMPLE



Notice that matrix multiplication is not commutative!

$AB \neq BA$

ANOTHER EXAMPLE



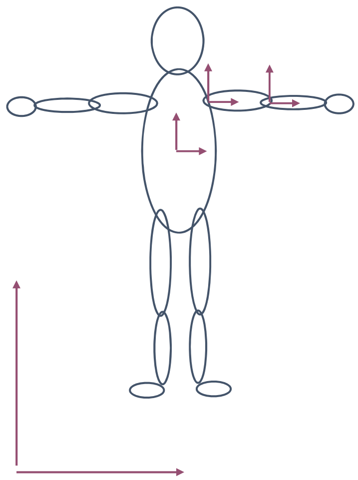
Notice that matrix multiplication is not commutative!
$AB \neq BA$

$$M_{21} = Tr_{(2,1)} \cdot Rot_{\pi/4} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Rot_{\pi/4} \cdot Tr_{(2,1)} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & \mathbf{1/\sqrt{2}} \\ 1/\sqrt{2} & 1/\sqrt{2} & \mathbf{3/\sqrt{2}} \\ 0 & 0 & 1 \end{pmatrix}$$

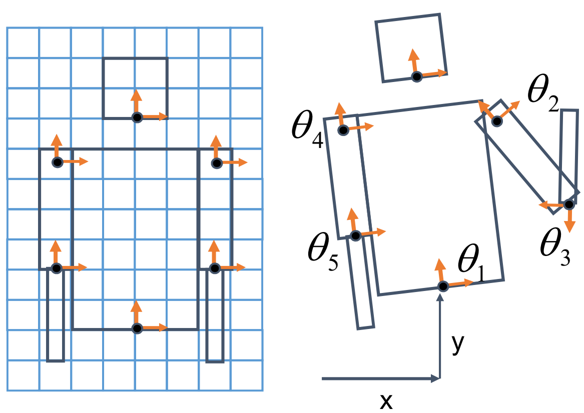
TRANSFORMATION HIERARCHIES

TRANSFORMATION HIERARCHIES

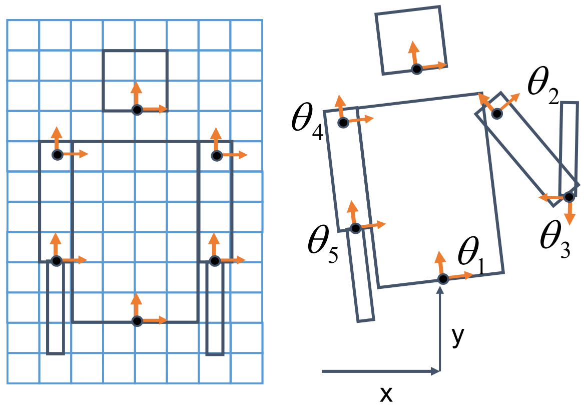


- Scenes have multiple coordinate systems
  - Often strongly related
    - Parts of the body
    - Object on top of each other
      - Next to each other...
- Independent definition is bug prone
- Solution: Transformation Hierarchies

TRANSFORMATION HIERARCHY EXAMPLES



TRANSFORMATION HIERARCHY EXAMPLES

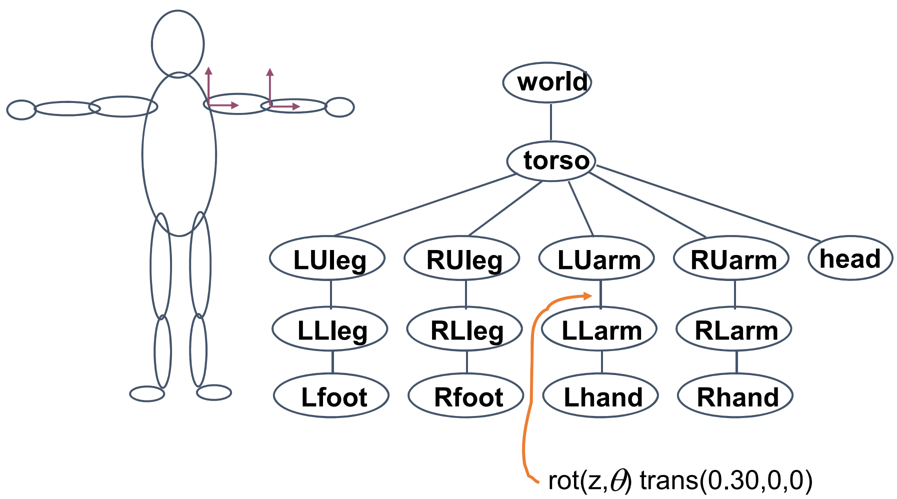


$M_1 = Tr_{(x,y)} \cdot Rot\theta_1$

$M_2 = M_1 \cdot Tr_{(2.5,5.5)} \cdot Rot\ \theta_2$

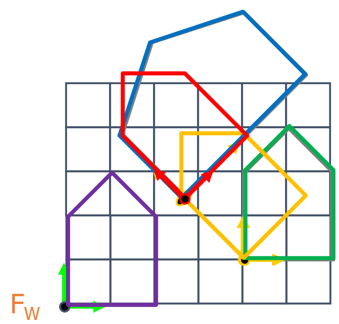
$M_3 = M_2 \cdot Tr_{(0,-3.5)} \cdot Rot\ \theta_3$

TRANSFORMATION HIERARCHIES



TRANSFORMATION HIERARCHY QUIZ

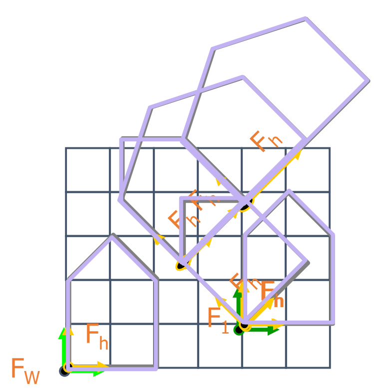
```
M.setIdentity();
M = M*Translation(4,1,0);
M = M*Rotation(pi/4,0,0,1);
House.matrix = M;
```



- Which color house will we draw?
  A. Red
  B. Blue
  C. Green
  D. Orange
  E. Purple

TRANSFORMATION HIERARCHY EXAMPLES

```
   M.setIdentity();
1) House1.matrix = M;

2) M = M*Translation(4,1,0);
   House2.matrix = M;

3) M = M*Rotation(pi/4,0,0,1);
   House3.matrix = M;

4) M = M*Translation(0,2,0);
   House4.matrix = M;

5) M = M*Scale(2,1,1);
   House5.matrix = M;

6) M = M*Translation(1,0,0);
   House6.matrix = M;
```



HIERARCHICAL MODELING

- Advantages
  - Define object once, instantiate multiple copies
  - Transformation parameters often good control knobs
  - Maintain structural constraints if well-designed
- Limitations
  - Expressivity: not always the best controls
  - Can’t do closed kinematic chains
    - Keep hand on hip

WHAT HAPPENS IF…

- We do multiple translations one after another?
- Multiple scales?
- Multiple rotations?

BREAKING TRANSFORMATIONS INTO CHUNKS

- How do we reflect around arbitrary line (2D)?
  - Rotate/translate line to a known axis
- How do we reflect around arbitrary plane (3D)?
  - Rotate/translate plane to a known plane
- How to rotate around arbitrary axis (3D)?
  - Rotate axis to known axis
- Derived on whiteboard…