

CPSC 314
07 - AFFINE
TRANSFORMATIONS

TEXTBOOK: I.3

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TRANSFORMATIONS WE WANT

The diagram illustrates six types of affine transformations applied to a letter 'S' on a 2D coordinate system:

- Scaling:** A green 'S' is shown within a dashed box, indicating its original size and position.
- Translation:** A light blue 'S' is shown shifted to a new position.
- Rotation:** A light blue 'S' is shown rotated around its center.
- Reflection:** A light blue 'S' is shown mirrored across a vertical axis.
- Reflection:** A light blue 'S' is shown mirrored across a vertical axis and rotated.
- Shear:** A light blue 'S' is shown skewed, where the vertical lines are no longer parallel to the y-axis.

TRANSLATION

- There's a minor glitch.
- Translation:

$$\begin{aligned}L: u &\rightarrow u + b \\L(0) &= b \neq 0\end{aligned}$$

Translation is not a linear transformation and can't be represented as a matrix operation in this space.

TRANSLATION

- There's a minor glitch.
- Translation:

$$\begin{aligned}L: u &\rightarrow u + b \\L(0) &= b \neq 0\end{aligned}$$

We need more power.

GENERAL TRANSFORMATIONS

We need to represent all the
linear transformations + translation.

Ideas?

GENERAL TRANSFORMATIONS

We need to represent all the
linear transformations + translation.

Ideas?

$$T(\mathbf{v}) = M\mathbf{v} + \mathbf{b}$$

COMPOSITING SNEAK PEEK

- How do we first rotate, then scale?

- How do we first rotate, then translate?

UNIFYING

- We want something more universal:

$$T(v) = Mv$$

Then we could easily combine different transformations or invert

Let's extend the vector to have one artificial coordinate: $(x,y,1)$

Then what should the matrix be?

AUGMENTED MATRIX

$$M_{2 \times 2} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} M_{2 \times 2} & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

Haven't changed much, have we?

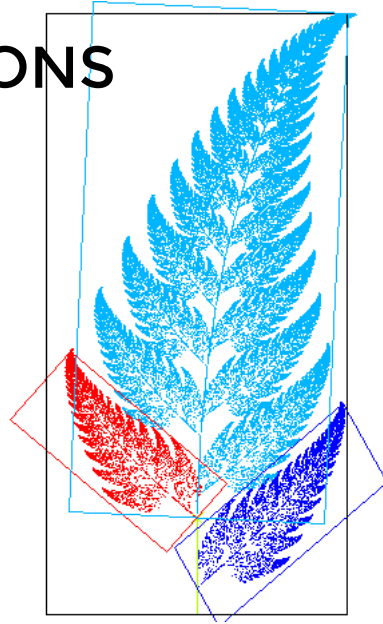
AUGMENTED MATRIX

$$\begin{bmatrix} M_{2 \times 2} & \mathbf{b}_x \\ & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' + b_x \\ y' + b_y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} + \mathbf{b}$$

Translation 

AFFINE TRANSFORMATIONS

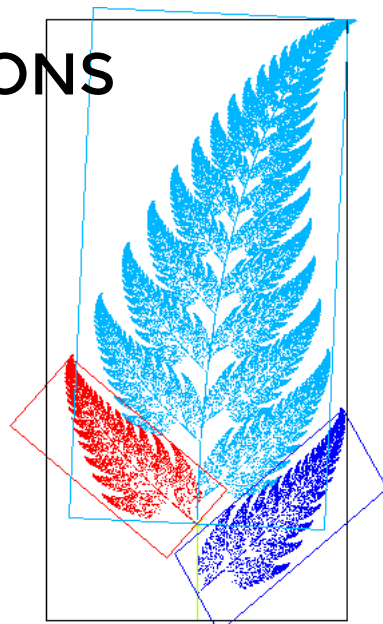
Linear (rotation, scaling, shear, reflections)
+ TRANSLATION



AFFINE TRANSFORMATIONS

Linear (rotation, scaling, shear, reflections)
+ TRANSLATION

How to convert a linear transformation
matrix into affine matrix?



AFFINE TRANSFORMATIONS

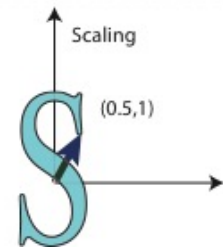
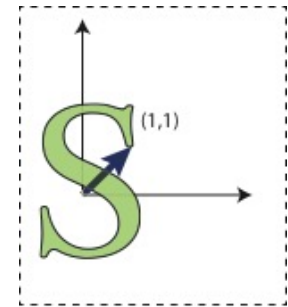
- Scale:

$$M = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

$$M = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example:

$$\begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ 2\beta \\ 3\gamma \\ 1 \end{pmatrix}$$



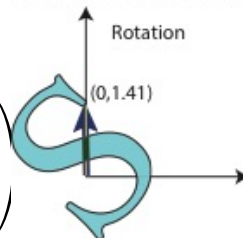
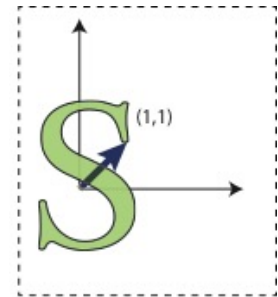
AFFINE TRANSFORMATIONS

- Rotation

$$M = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example:

$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) - \sin(\alpha) \\ \cos(\alpha) + \sin(\alpha) \\ 0 \\ 1 \end{pmatrix}$$



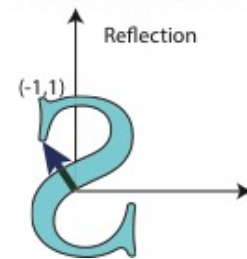
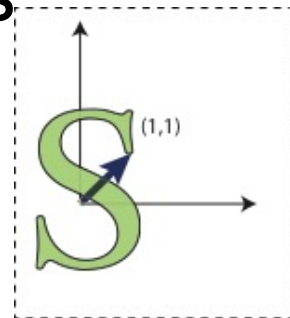
AFFINE TRANSFORMATIONS

- Reflection (y)

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example:

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \\ 1 \end{pmatrix}$$



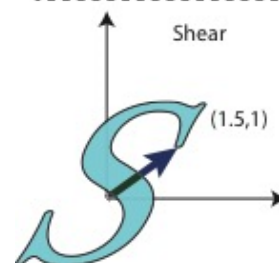
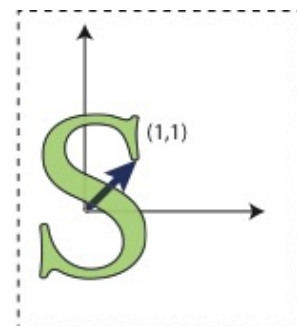
AFFINE TRANSFORMATIONS

- Shear (along x)

$$M = \begin{bmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + \lambda y \\ y \\ z \\ 1 \end{pmatrix}$$



AFFINE TRANSFORMATIONS

- Translation

$$M = \begin{bmatrix} 1 & 0 & 0 & c_x \\ 0 & 1 & 0 & c_y \\ 0 & 0 & 1 & c_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 0 & 0 & c_x \\ 0 & 1 & 0 & c_y \\ 0 & 0 & 1 & c_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + c_x \\ y + c_y \\ z + c_z \\ 1 \end{pmatrix}$$

AFFINE TRANSFORMATIONS: 2D

$$M = \begin{bmatrix} 1 & 0 & 0 & c_x \\ 0 & 1 & 0 & c_y \\ 0 & 0 & 1 & c_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

AFFINE TRANSFORMATIONS: 2D

$$M = \begin{bmatrix} 1 & 0 & c_x \\ 0 & 1 & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

WHAT'S THAT 1?

- Now our points are represented by $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ - homogeneous coordinates
- Those are vectors from ANOTHER SPACE.