1. Transformations as a change of coordinate frame

NOTE: We will consider all coordinates as tuples of (i, j)

(a) (1 point) Express the coordinates of point P with respect to coordinate frames A, B, and C.

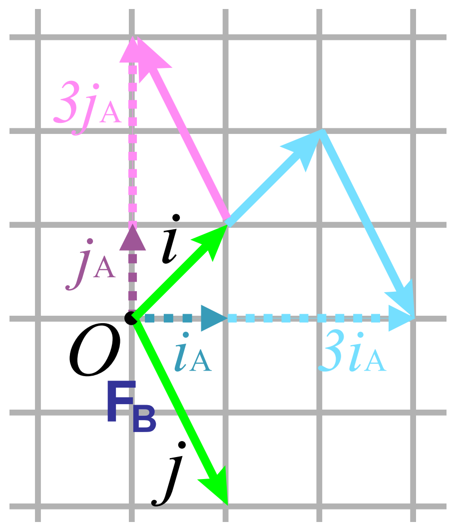
$$P = \begin{pmatrix} 4 \\ 3 \end{pmatrix}_A = \begin{pmatrix} 1/3 \\ 5/3 \end{pmatrix}_B = \begin{pmatrix} 3/2 \\ -1/4 \end{pmatrix}_C$$

Consider finding the clean unit vectors i_A *and* j_A *from every coordinate frame. If we had this, we can easily get to our point P from any origin:*

$$\begin{aligned} P &= O_A + 4\mathbf{i}_A + 3\mathbf{j}_A & (1)\\ &= O_B + 2\mathbf{i}_A + (-3)\mathbf{j}_A & (2)\\ &= O_C + 2\mathbf{i}_A + 1\mathbf{j}_A & (3) \end{aligned}$$

Now look at frame B. We can get a horizontal vector with $2\mathbf{i}_B + 1\mathbf{j}_B = \binom{2}{1}_B$*. We can turn this into our unit vector* i_A *by scaling by 1/3:*

$$i_A = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}_B = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}_B$$



Similarly, we can get our vertical unit vector j_A*:*

$$j_A = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix}_B = \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}_B$$

Now we can substitute this into (2)*, giving us:*

$$\begin{aligned} P &= O_B + 2\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}_B + (-3)\begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}_B \\ &= O_B + (1/3)\mathbf{i}_B + (5/3)\mathbf{j}_B \\ &= \begin{pmatrix} 1/3 \\ 5/3 \end{pmatrix}_B \end{aligned}$$

We do the same for frame C:

$$i_A = \frac{1}{4}\begin{pmatrix} 2 \\ -1 \end{pmatrix}_C = \begin{pmatrix} 2/4 \\ -1/4 \end{pmatrix}_C$$

$$j_A = \frac{1}{4}\begin{pmatrix} 2 \\ 1 \end{pmatrix}_C = \begin{pmatrix} 2/4 \\ 1/4 \end{pmatrix}_C$$

Substituting into (3)*:*

$$\begin{aligned} P &= O_C + 2\begin{pmatrix} 2/4 \\ -1/4 \end{pmatrix}_C + 1\begin{pmatrix} 2/4 \\ 1/4 \end{pmatrix}_C \\ &= O_C + (6/4)\mathbf{i}_C + (-1/4)\mathbf{j}_C \\ &= \begin{pmatrix} 3/2 \\ -1/4 \end{pmatrix}_C \end{aligned}$$

(b) (1 point) Express the coordinates of vector V with respect to coordinate frames A, B, and C.

$$\mathbf{V}_A = (-1)\mathbf{i}_A + (-2)\mathbf{j}_A = \begin{pmatrix} -1 \\ -2 \end{pmatrix}_A$$

We've done most of the work already in part (a).

$$\begin{aligned} \mathbf{V}_B &= (-1)\mathbf{i}_A + (-2)\mathbf{j}_A \\ &= (-1)\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}_B + (-2)\begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}_B \\ &= \begin{pmatrix} -4/3 \\ 1/3 \end{pmatrix}_B \end{aligned}$$

$$\begin{aligned} \mathbf{V}_C &= (-1)\mathbf{i}_A + (-2)\mathbf{j}_A \\ &= (-1)\begin{pmatrix} 2/4 \\ -1/4 \end{pmatrix}_C + (-2)\begin{pmatrix} 2/4 \\ 1/4 \end{pmatrix}_C \\ &= \begin{pmatrix} -3/2 \\ -1/4 \end{pmatrix}_C \end{aligned}$$

Presented here is the alternate "parallel" method for finding $\mathbf{V}_B$*: This time, we want to find the intercept of* $\mathbf{j}$ *and* ℓ*. Define*

$$\ell := \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\mathbf{j} := \beta \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \text{we can see graphically that } \beta > 0.$$

Then we solve,

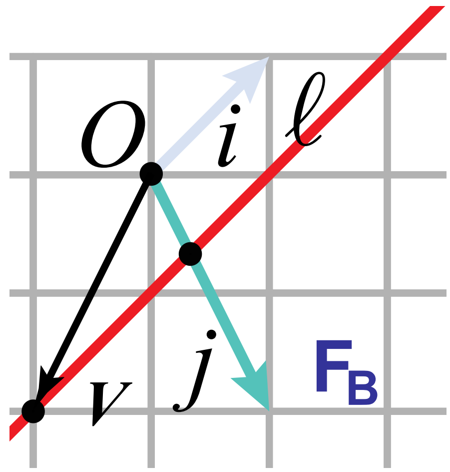
$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \beta \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\implies \begin{cases} \alpha = \beta \\ \alpha + 2\beta = 1 \end{cases}$$

$$\implies 3\beta = 1$$

$$\implies \beta = \frac{1}{3}$$



We find the length from $(-1, -2)$ *to the intersection to determine the* $\mathbf{i}$ *component*

$$\left\| \frac{1}{3}\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} 1/3 \\ -2/3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\|_2$$

$$= \left\| \begin{pmatrix} 4/3 \\ 4/3 \end{pmatrix} \right\|_2$$

$$= \sqrt{\left(\tfrac{4}{3}\right)^2 + \left(\tfrac{4}{3}\right)^2} = \sqrt{\frac{32}{9}} = \frac{4\sqrt{2}}{3}$$

Since we know $\|\mathbf{i}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$*, we divide this distance by* $\mathbf{i}$*'s length to find the contribution of* $\mathbf{i}$

$$\frac{4\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} = \frac{4}{3}$$

Thus, we get $\mathbf{V}_B = \left(-\frac{4}{3}, \frac{1}{3}\right)$

(c) (2 points) Fill in the 2D transformation matrix that takes points from F_C to F_A, as given to the right of the above figure.

To form the matrix that converts points in F_C to F_A, we express the basis vectors $\mathbf{i}$ *and* $\mathbf{j}$ *of F_C and the translation of F_C's origin in terms of F_A coordinates.*

$$\mathbf{i}_C = 1\mathbf{i}_A + 1\mathbf{j}_A$$
$$\mathbf{j}_C = -2\mathbf{i}_A + 2\mathbf{j}_A$$
$$O_C = O_A + 2\mathbf{i}_A + 2\mathbf{j}_A$$

$$\begin{bmatrix} i_1 & j_1 & t_1 \\ i_2 & j_2 & t_2 \\ 0 & 0 & 1 \end{bmatrix}_{C\to A} = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

To verify, we use the point we computed from part (a),

$$\begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3/2 \\ -1/4 \\ 1 \end{bmatrix}_C = \begin{bmatrix} 3/2 + 2/4 + 2 \\ 3/2 - 2/4 + 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/2 + 2 \\ 2/2 + 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}_A$$

(d) (2 points) Fill in the 2D transformation matrix that takes points from F_A to F_B, as given to the right of the above figure.

One way to do this is to find the inverse matrix that converts points from F_B to F_A (which is nice), and then invert it (which is not so nice).

However, we already did all the work in part (a) that expresses F_A's basis in terms of F_B, so let's just express F_A's origin in terms of F_B to directly obtain our matrix. From part (a):

$$i_A = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}_B$$
$$j_A = \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}_B$$

To find O_A in terms of F_B, we use the same method as part (a):

$$\begin{aligned} O_A &= O_B + (-2)\mathbf{i}_A + (-6)\mathbf{j}_A \\ &= O_B + (-2)\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}_B + (-6)\begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}_B \\ &= O_B + (-10/3)\mathbf{i}_B + (4/3)\mathbf{j}_B \\ &= \begin{pmatrix} -10/3 \\ 4/3 \end{pmatrix}_B \end{aligned}$$

$$\begin{bmatrix} i_1 & j_1 & t_1 \\ i_2 & j_2 & t_2 \\ 0 & 0 & 1 \end{bmatrix}_{A \to B} = \begin{bmatrix} 2/3 & 1/3 & -10/3 \\ 1/3 & -1/3 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}$$

To verify, we use the point we computed from part (a),

$$\begin{bmatrix} 2/3 & 1/3 & -10/3 \\ 1/3 & -1/3 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}_A = \begin{bmatrix} 8/3 + 3/3 - 10/3 \\ 4/3 - 3/3 + 4/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 5/3 \\ 1 \end{bmatrix}_B$$

Presented below is the alternative method, where we find the matrix converting points from F_B to F_A and invert it.

$$\begin{bmatrix} i_1 & j_1 & t_1 \\ i_2 & j_2 & t_2 \\ 0 & 0 & 1 \end{bmatrix}_{B \to A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & -2 & 6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -2 \\ 1 & -2 & 0 & 0 & 1 & -6 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 2 & 1 & -10 \\ 1 & -2 & 0 & 0 & 1 & -6 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 1/3 & -10/3 \\ -1 & 2 & 0 & 0 & -1 & 6 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 1/3 & -10/3 \\ 0 & 2 & 0 & 2/3 & -2/3 & 8/3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 1/3 & -10/3 \\ 0 & 1 & 0 & 1/3 & -1/3 & 4/3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

The right side matrix is exactly the one we found with the direct method:

$$\begin{bmatrix} 2/3 & 1/3 & -10/3 \\ 1/3 & -1/3 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}_{A \to B}$$

(e) (2 points) Using the above two matrices, develop a 2D transformation matrix that takes points from F_C to F_B. Test your solution using point P.

Given the matrices from parts (c) and (d), we have $M_{C\to A}$, the matrix that converts points from F_C to F_A, and $M_{A\to B}$, the matrix that converts points from F_A to F_B. We notice that since $M_{C\to A}P_C$ gives a point in F_A from a point in F_C, we can use that point to compute the same point in F_B.

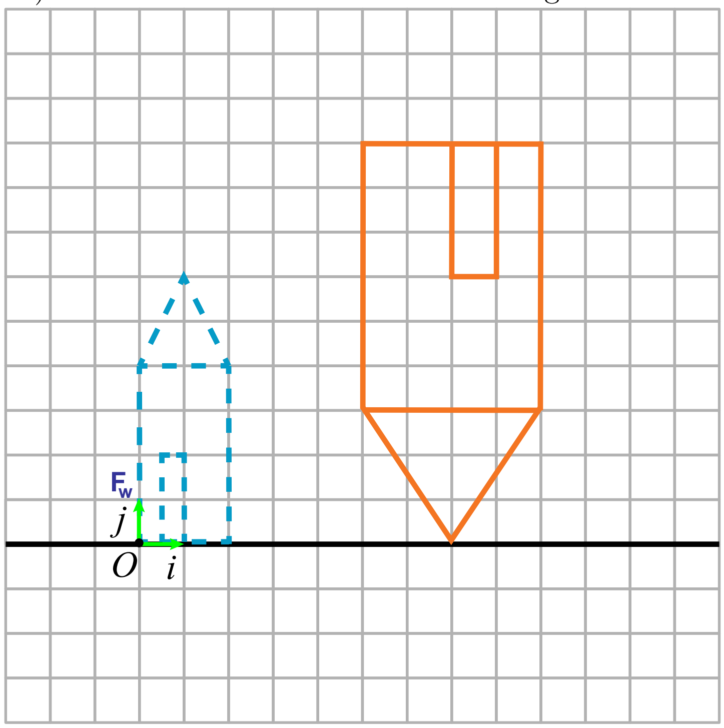
$$\begin{aligned}
P_A &= M_{C\to A}P_C \\
P_B &= M_{A\to B}P_A \\
&= M_{A\to B}\big[M_{C\to A}P_C\big] \\
\implies P_B &= M_{C\to B}P_C \\
\implies M_{C\to B} &= M_{A\to B}M_{C\to A} \\
&= \begin{bmatrix} 2/3 & 1/3 & -10/3 \\ 1/3 & -1/3 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\
M_{C\to B} &= \begin{bmatrix} 1 & -2/3 & -4/3 \\ 0 & -4/3 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

To verify,

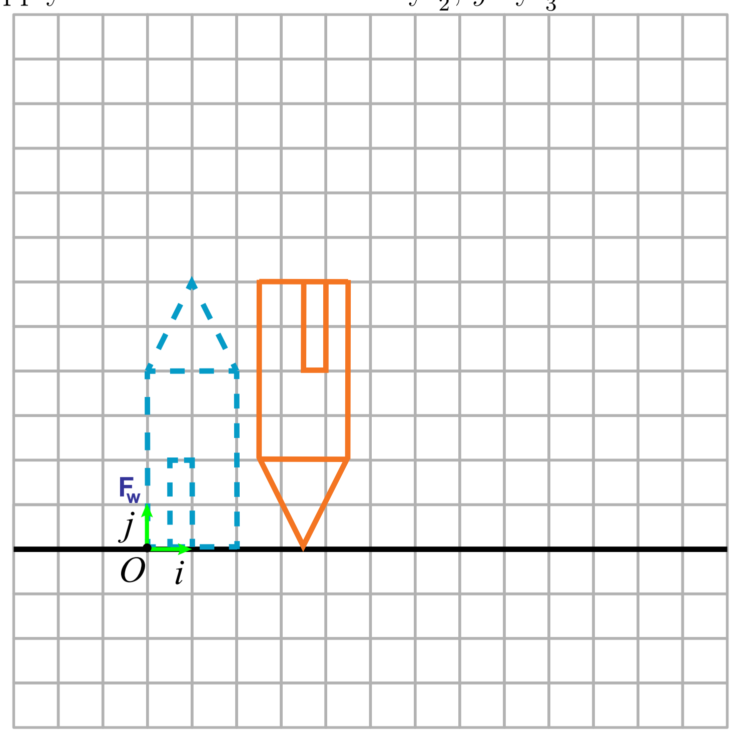
$$\begin{aligned}
\begin{bmatrix} 1 & -2/3 & -4/3 \\ 0 & -4/3 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3/2 \\ -1/4 \\ 1 \end{bmatrix}_C &= \begin{bmatrix} 3/2 + 2/12 - 4/3 \\ 4/12 + 4/3 \\ 1 \end{bmatrix}_B \\
&= \begin{bmatrix} 9/6 + 1/6 - 8/6 \\ 1/3 + 4/3 \\ 1 \end{bmatrix}_B = \begin{bmatrix} 1/3 \\ 5/3 \\ 1 \end{bmatrix}_B
\end{aligned}$$

2. Composing Transformations

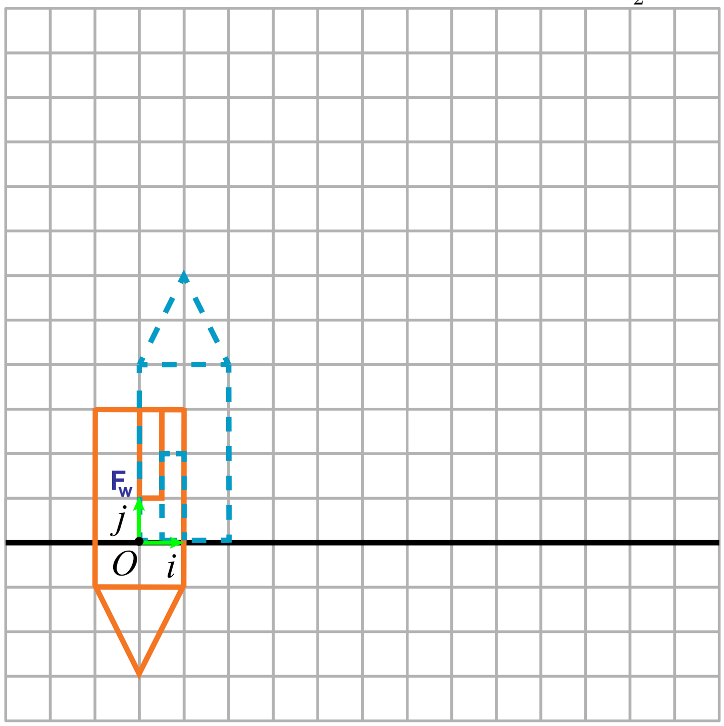
(a) (2 points) • We start with the house looking like this



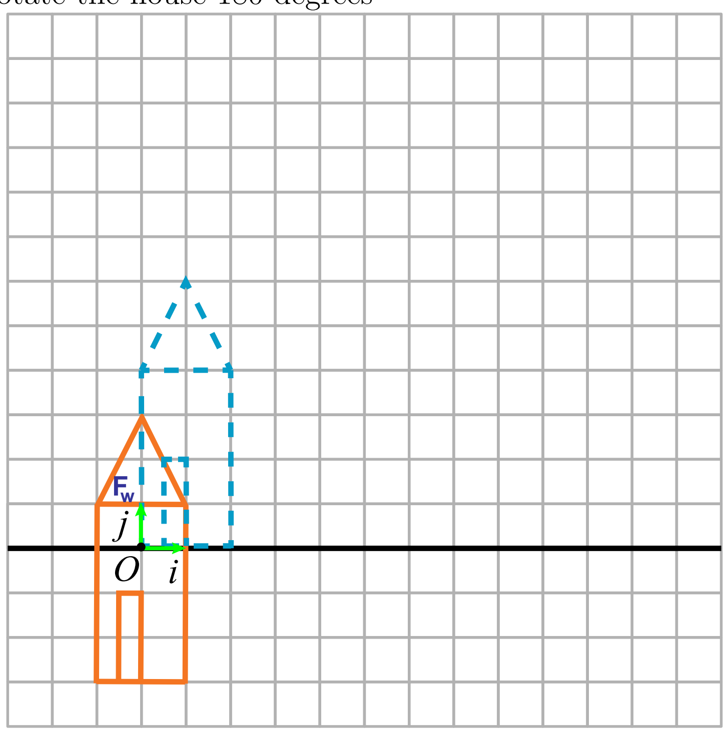
• Apply a non-uniform scale: x by $\frac{1}{2}$, y by $\frac{2}{3}$



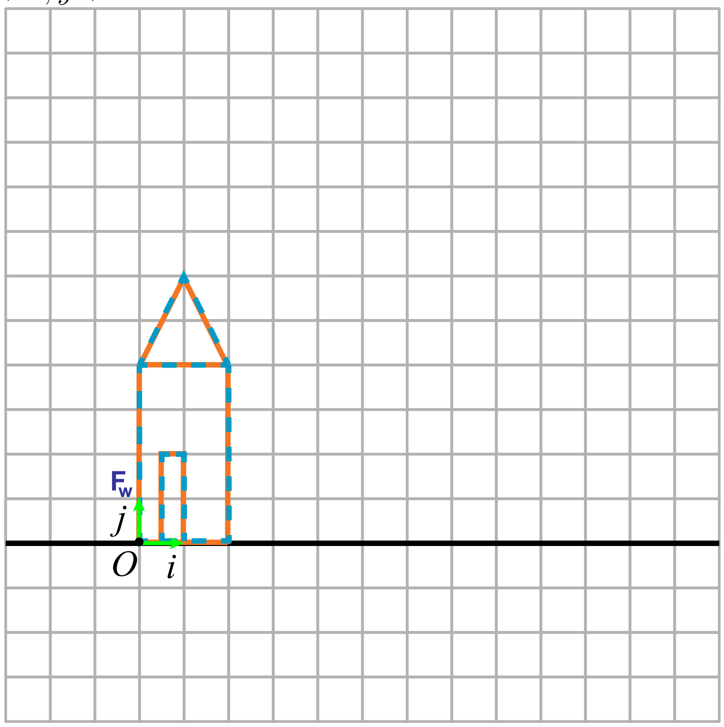
- Translate the house so that the centre is at the origin of the world coordinate frame. That is, we perform a translation of $x - \frac{7}{2}$, $y - 3$



- Rotate the house 180 degrees



- Finally, translate the house to match the dashed outline using a translation of $x+1, y+3$



(b) (2 points) Give the resulting 4×4 transformation matrix. Assume that the transformation leaves z to be unaltered.

$$\begin{bmatrix}1&0&0&1\\0&1&0&3\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&-7/2\\0&1&0&-3\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1/2&0&0&0\\0&2/3&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}-1/2&0&0&9/2\\0&-2/3&0&6\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

(c) (2 points) What values would need to be assigned to `theta, a, b, c, i, j, k, x, y, z` in order for the following transformations to yield an identical final transformation?

Note: `THREE.Matrix4()` *constructs an identity matrix. Also note that here we pretend that * does matrix multiplication. In actual JS code, you'd use multiplyMatrices() function instead. Also, we combine our two translation matrices together - moving the interior translation left of the rotate flips the signs of* $-7/2$ *and* -3.

```
var m = new THREE.Matrix4();
var m1 = new THREE.Matrix4();
var m2 = new THREE.Matrix4();
var m3 = new THREE.Matrix4();
m1.makeRotateAxis(new THREE.Vector3(i=0,j=0,k=1), theta=180);
m2.makeScale(a=0.5,b=(2.0/3.0),c=1.0);
m3.makeTranslate(x=4.5,y=6.0,z=0.0);
m = m3*m2*m1;
house.geometry.applyMatrix(m);
```

3. Decompose the following complex transformations in homogeneous coordinates into a product of simple transformations (scaling, rotation, translation, shear). Pay attention to the order of transformations.

   (a) (1 point)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Shear on y:} \\ y=y-x}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{CCW rotation of 90 degrees} \\ \text{on the X-axis}}}$$

   (b) (1 point)

$$\begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 0.2 & 0 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Translation}} \cdot \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Non-uniform scaling}}$$

   (c) (2 points) What are the inverses of the matrices of parts (a) and (b) above?
   Hint: if $M = AB$, then $M^{-1} = B^{-1}A^{-1}$

   Part A: *It is easy to think of the inverse of some rotation of θ degrees as the rotation about the same axis of $-\theta$ degrees. Since we have a very simple shear $y = y - x$, we can invert this shear by applying $y = y + x$. Then using the hint, we find*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{CW rotation of 90 degrees} \\ \text{on the X-axis}}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Shear on y:} \\ y=y+x}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   Further note: it is useful to recall that sin *is an odd function and* cos *is an even function. Thus, the inversions become a flip of the sign for entries with* sin, *but the entries with* cos *do not change signs. In fact, the inverse of any rotation matrix is simply its transpose.*

   Part B:

$$\begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 0.2 & 0 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 & 1.5 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d) (2 points) Give the sequence of THREE.js transformations that would produce the same transformation matrix as in part (a) of this question.
Note: THREE.js matrix indexing is column-wise, but initialization is row-wise! So no transposing needed here for hand-initializing our shear matrix `m1`.
See ***https://threejs.org/docs/api/math/Matrix4.html***.

```
var m = new THREE.Matrix4();
var m1 = new THREE.Matrix4();
var m2 = new THREE.Matrix4();
m1.set(1, 0, 0, 0,
      -1, 1, 0, 0,
       0, 0, 1, 0,
       0, 0, 0, 1);
m2.makeRotateAxis(new THREE.Vector3(1,0,0), 90);
m = m1*m2;
```