

# CPSC 314

## Theory Assignment 1

Due 4PM Sep 23,2016

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Name: \_\_\_\_\_

Student Number: \_\_\_\_\_

Question 0	/ 0
Question 1	/ 1
Question 2	/ 1
Question 3	/ 1
Question 4	/ 1
Question 5	/ 1
TOTAL	/ 5

## 1. (1 point) Vectors

$$a = \begin{pmatrix} 2 \\ -5 \\ 8 \end{pmatrix} \quad b = \begin{pmatrix} -3 \\ 6 \\ 8 \end{pmatrix}$$

- compute  $a \cdot b$ ,

$$a \cdot b = 2 \cdot (-3) + (-5) \cdot 6 + 8 \cdot 8 = 28$$

- compute  $a^T b$

*Following the rules of matrix multiplication,*

$$a^T b = \begin{pmatrix} 2 & -5 & 8 \end{pmatrix} \begin{pmatrix} -3 \\ 6 \\ 8 \end{pmatrix} = a \cdot b = 28$$

- compute  $a \times b$

*One of the ways of calculating cross products is via a matrix determinant:*

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 8 \\ -3 & 6 & 8 \end{vmatrix} = \mathbf{i} \cdot (-5 \cdot 8 - 8 \cdot 6) - \mathbf{j} \cdot (2 \cdot 8 - 8 \cdot (-3)) + \mathbf{k} \cdot (2 \cdot 6 - (-5) \cdot (-3)) =$$

$$= -88\mathbf{i} - 40\mathbf{j} - 3\mathbf{k} = \begin{pmatrix} -88 \\ -40 \\ -3 \end{pmatrix},$$

*where  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are unit vectors.*

- compute  $b \times a$

$$b \times a = -a \times b = \begin{pmatrix} 88 \\ 40 \\ 3 \end{pmatrix}$$

## 2. (1 point) Matrices

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 5 \\ -9 & 7 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 1 & 5 \\ 1 & 1 & 0 \\ 9 & -1 & 1 \end{pmatrix}$$

- Compute  $C = AB$ .

$$\begin{pmatrix} 28 & -4 & 8 \\ 49 & -3 & 10 \\ 7 & -5 & -42 \end{pmatrix}$$

- Does  $AB = BA$ ?

*Though in some specific cases matrix multiplication is commutative, generally it is not. Here, if you calculate  $BA$  you will see that it is different from  $AB$ :*

$$BA = \begin{pmatrix} -41 & 30 & 29 \\ 2 & -1 & 8 \\ -1 & -12 & 25 \end{pmatrix}$$

- Given the vector  $a$  from the previous question, compute  $c = Aa$ .

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 2 & 5 \\ -9 & 7 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 + (-2) \cdot (-5) + 3 \cdot 8 \\ 1 \cdot 2 + 1 \cdot (-5) + 5 \cdot 8 \\ (-9) \cdot 2 + 7 \cdot (-5) + 3 \cdot 8 \end{pmatrix} = \begin{pmatrix} 36 \\ 37 \\ -29 \end{pmatrix}$$

- Given the vector  $a$  from the previous question, compute  $d = a^T A$ .

$$\begin{aligned} & \begin{pmatrix} 2 & -5 & 8 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 5 \\ -9 & 7 & 3 \end{pmatrix} = \\ & = \begin{pmatrix} 2 \cdot 1 - 5 \cdot 1 + 8 \cdot (-9) & 2 \cdot (-2) - 5 \cdot 1 + 8 \cdot 7 & 2 \cdot 3 - 5 \cdot 5 + 8 \cdot 3 \end{pmatrix} = \\ & = \begin{pmatrix} -75 & 47 & 5 \end{pmatrix} \end{aligned}$$

## 3. (1 point) Normals and Planes

$T$  is a triangle in 3D with vertices  $P_1 = (1, 1, 0)$ ,  $P_2 = (1, 0, 1)$  and  $P_3 = (3, 1, 0)$  (counterclockwise around the normal).

- Compute the normal to  $T$ .

*We know that cross product of two vectors in a plane is orthogonal to that plane. Therefore, we can use the normalized cross product as the unit normal to the plane:*

$$\begin{aligned}\overrightarrow{P_1P_2} &= (0, -1, 1); \overrightarrow{P_1P_3} = (2, 0, 0) \\ \vec{n} &= \frac{\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}}{\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\|} = \left(\frac{0}{\sqrt{8}}, \frac{2}{\sqrt{8}}, \frac{2}{\sqrt{8}}\right)^T = \left(\frac{0}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T\end{aligned}$$

- Compute the area of  $T$ .

*One of the formulas to calculate the area of the triangle is that it is equal to half of the norm of the cross product of the triangle's sides, which we have already computed for the previous answer:*

$$S_T = \frac{1}{2} \|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\| = \frac{1}{2} \sqrt{8} = \sqrt{2}$$

- Compute the implicit plane equation  $Ax + By + Cz + D = 0$  for the plane that  $T$  lies in.

*The plane equation is not unique: we choose all the coefficients  $A, B, C, D$  up to a constant factor. We know that  $(A, B, C)$  is collinear to  $\vec{n}$ . For convenience, we multiply the normal by  $\sqrt{2}$  to get rid of the ugly denominator, we can take*

$$(A, B, C) = (0, 1, 1)$$

*To calculate  $D$ , we use the fact that the plane should go through each of  $P_1, P_2, P_3$ . Let's use, for example,  $P_1$ :*

$$0 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 + D = 0$$

$$D = -1$$

## 4. (1 point) Segments and Lines

Given two segments in 2D:  $S_1$  from  $(0, 1)$  to  $(-1, 3)$  and  $S_2$  from  $(0, 0)$  to  $(3, 3)$ ,

- Does the point  $P = (0, 1)$  lie on  $S_1$ ?

*Let's presume that  $P = (P_x, P_y)$  lies on  $S_1$  and denote endpoints of  $S_1$  by  $A$  and  $B$ . Then  $\overrightarrow{AP}$  is collinear to  $\overrightarrow{AB}$ , i.e.  $\overrightarrow{AP} = \alpha \overrightarrow{AB}$  for some  $0 \leq \alpha \leq 1$ . So, we would write system of equations like that:*

$$\{ P_x - 0 = \alpha \cdot (-1 - 0)$$

$$\{ P_y - 1 = \alpha \cdot (3 - 1)$$

If  $P$  lies on the line connecting  $A$  and  $B$ , those two equations are linearly dependent (i.e. the same up to a constant factor), then there exists one  $\alpha$  satisfying both of them. We calculate this alpha by solving either of the equations and check if it lies within the  $[0, 1]$  range - that means that the point is inside the segment  $S_1$ .

If we plug in  $P = (0, 1)$  we'll realize that there is no  $\alpha$  that will satisfy the equations, thus  $P$  does not lie on  $S_1$

- Do  $S_1$  and  $S_2$  intersect? If yes, compute the intersection, if no, explain. Let's denote  $S_1 = \overrightarrow{AB}$  and  $S_2 = \overrightarrow{CD}$ . As we know, any point on segment  $S_1$  can be expressed as  $A + \alpha \cdot \overrightarrow{AB}$  for some  $\alpha \in [0, 1]$ . Similarly, any point on  $S_2$  can be expressed as  $C + \beta \cdot \overrightarrow{CD}$  for some  $\beta \in [0, 1]$ . If those two segments intersect, there must exist one point  $E = A + \alpha \cdot \overrightarrow{AB} = C + \beta \cdot \overrightarrow{CD}$ . So, we can write down a system of equations:

$$\{ A_x + \alpha \cdot (B_x - A_x) = C_x + \beta \cdot (D_x - C_x)$$

$$\{ A_y + \alpha \cdot (B_y - A_y) = C_y + \beta \cdot (D_y - C_y),$$

solve for  $\alpha$  and  $\beta$ , and check if both of them are within  $[0, 1]$  range.

In our case, the system looks like:

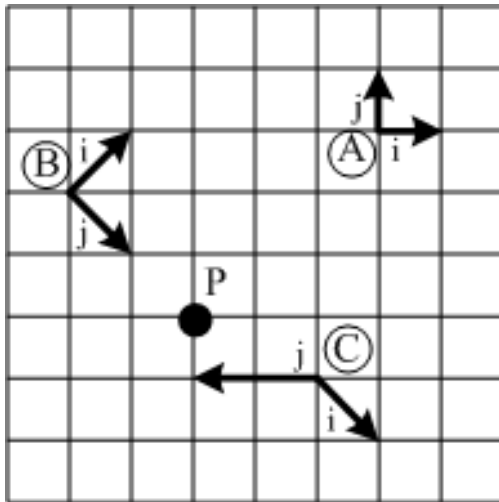
$$\{ -\alpha = 4\beta$$

$$\{ 1 + 2\alpha = 3\beta,$$

So  $\alpha = -\frac{4}{11}$  and  $\beta = \frac{1}{11}$ . As  $\alpha \notin [0, 1]$ , the final answer is no, those segment don't intersect.

## 5. (1 point) Frames

Specify the coordinates of point  $P$  with respect to coordinate frames A, B and C, and then explain the coordinate transformation process from  $P_A$  to  $P_B$ .



Let's use coordinate system A as our reference system. Then

$$\mathbf{i}_B = \mathbf{i}_A + \mathbf{j}_A,$$

$$\mathbf{j}_B = \mathbf{i}_A - \mathbf{j}_A,$$

$$O_B = O_A - 5\mathbf{i}_A - \mathbf{j}_A,$$

where  $O_A$  and  $O_B$  are the origins of their respective coordinate frames. Also,  $P = O_A - 3\mathbf{i}_A - 3\mathbf{j}_A$ . Now, as we want to find out expression of  $P$  using unit vectors of B only, we should express  $\mathbf{i}_A$  and  $\mathbf{j}_A$  using  $\mathbf{i}_B, \mathbf{j}_B$ , thus finding coordinates of A's unit vector in B coordinate system. Using the previous equations,

$$\mathbf{i}_A = \frac{1}{2} \cdot (\mathbf{i}_B + \mathbf{j}_B)$$

$$\mathbf{j}_A = \frac{1}{2} \cdot (\mathbf{i}_B - \mathbf{j}_B).$$

Also,  $O_A = O_B + 5\mathbf{i}_A + \mathbf{j}_A$ , so substituting it all to the expression for  $P$ , we get:

$$\begin{aligned} P &= O_A - 3\mathbf{i}_A - 3\mathbf{j}_A = O_B + 5\mathbf{i}_A + \mathbf{j}_A - 3\mathbf{i}_A - 3\mathbf{j}_A = \\ &= O_B + 2\mathbf{i}_A - 2\mathbf{j}_A = O_B + (\mathbf{i}_B + \mathbf{j}_B) - (\mathbf{i}_B - \mathbf{j}_B) = O_B + 2\mathbf{j}_B \end{aligned}$$

So,  $P_B = (0, 2)$ .

In coordinate frame A:  $P_A = (-3, -3)$

In coordinate frame B:  $P_B = (0, 2)$

In coordinate frame C:  $P_C = (-1, 0.5)$