CPSC 314
Assignment 2
— **SOLUTIONS** —

due: Wednesday, October 7, 2015, in class

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Name: _____

Student Number: _____

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1. Transformations as a change of coordinate frame

   NOTE: We will consider all coordinates as tuples of (i, j)

   (a) (1 point) Express the coordinates of point P with respect to coordinate frames A, B, and C.

   Since the coordinate frames A and B are sane, the solutions are trivial and can be read off from the grid:

$$F_A : P = (4, 4); \qquad F_B : P = (2, 0)$$

   For F_C however, we have to solve for a linear combination of its basis vectors. We know that

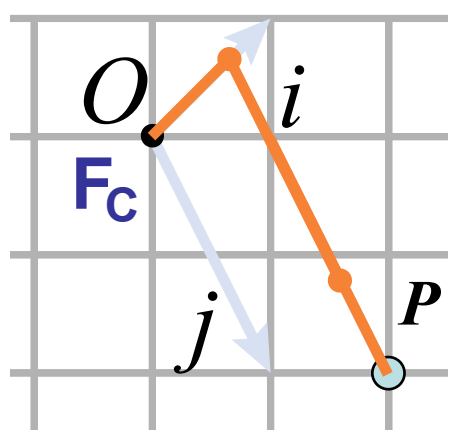
$$P = (2, -2) \qquad F_{cj} = (a, b) = (1, -2) \qquad F_{ci} = (c, d) = (1, 1)$$

   according to sanitized grid metrics, these are distances with respect to the origin in coordinate frame C. We set

$$P = \alpha\mathbf{i} + \beta\mathbf{j} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

   And then solve

$$\begin{aligned}
\implies & \begin{cases} \alpha a + \beta c = & 2 \\ \alpha b + \beta d = & -2 \end{cases} \\
\implies \alpha = \frac{2 - \beta c}{a} &= \frac{2 - \beta(1)}{(1)} = 2 - \beta \\
\implies (2 - \beta)b + \beta d &= -2 \\
2 - \beta - 2\beta &= -2 \\
-3\beta &= -4 \\
\beta &= \frac{4}{3} \\
\implies \alpha = 2 - \frac{4}{3} &= \frac{2}{3}
\end{aligned}$$



   We get the linear combination of $\mathbf{i}$ *and* $\mathbf{j}$ *that results in* $P = \left(\frac{2}{3}, \frac{4}{3}\right)$ *with respect to coordinate frame C. A good way to interpret this is the graphic to the right.*

(b) (1 point) Express the coordinates of vector V with respect to coordinate frames A, B, and C.

$$F_A : \mathbf{v} = (-1, -2)$$

$\mathbf{F_B}$: *Let's try another way of interpreting and solving this kind of problem. We want to find the multiple of* '$\mathbf{i}$' *that 'sets up'* $\mathbf{j}$ *on the line* ℓ*. In other words, we find the intercept of* $\mathbf{j}$ *and* ℓ*. Define*

$$\ell := \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\mathbf{i} := \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \text{we can see graphically that } \beta < 0.$$

Then we solve,

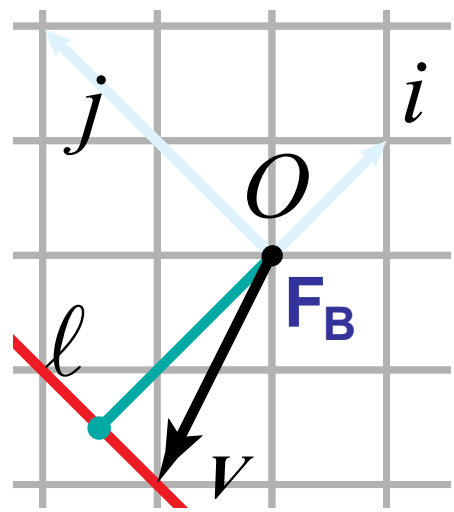
$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\implies \begin{cases} -\alpha = \beta \\ \alpha - 3 = \beta \end{cases}$$

$$\implies 2\beta = -3$$

$$\implies \beta = -\frac{3}{2}$$



Since we have constructed ℓ *to be parallel to* $\mathbf{j}$*, we simply have to find a (negative) multiple of* $\mathbf{i}$ *so that we reach* $(-1, -2)$ *from the intersection. To do this, we find the length*

$$\begin{aligned} \left\| -\frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right\|_2 &= \left\| \begin{pmatrix} -3/2 \\ -3/2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\|_2 \\ &= \left\| \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} \right\|_2 \\ &= \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} = \frac{1}{\sqrt{2}} \end{aligned}$$

Since the length of $\mathbf{j}$ *is* $\|\mathbf{j}\| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$*, to find the contribution of* $\mathbf{j}$*, we divide the distance from the intercept*

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{2}} = \frac{1}{4}$$

Thus, we get $\mathbf{v} = \left(-\frac{3}{2}, -\frac{1}{4}\right)$

$\mathbf{F_C}$: *This time, we want to find the intercept of* $\mathbf{j}$ *and* ℓ. *Define*

$$\ell := \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\mathbf{j} := \beta \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \text{we can see graphically that } \beta > 0.$$

Then we solve,

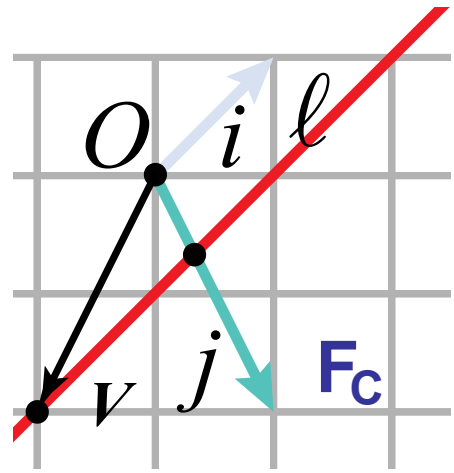
$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \beta \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\implies \begin{cases} \alpha = \beta \\ \alpha + 2\beta = 1 \end{cases}$$

$$\implies 3\beta = 1$$

$$\implies \beta = \frac{1}{3}$$



We find the length from $(-1, -2)$ *to the intersection to determine the* $\mathbf{i}$ *component*

$$\begin{aligned} \left\| \frac{1}{3} \begin{pmatrix} 1 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right\|_2 &= \left\| \begin{pmatrix} 1/3 \\ -2/3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\|_2 \\ &= \left\| \begin{pmatrix} 4/3 \\ 4/3 \end{pmatrix} \right\|_2 \\ &= \sqrt{\left(\tfrac{4}{3}\right)^2 + \left(\tfrac{4}{3}\right)^2} = \sqrt{\frac{32}{9}} = \frac{4\sqrt{2}}{3} \end{aligned}$$

Since we know $\|\mathbf{i}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$, *to find the contribution of* $\mathbf{i}$, *we divide the distance from the intercept*

$$\frac{4\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} = \frac{4}{3}$$

Thus, we get $\mathbf{v} = \left(-\frac{4}{3}, \frac{1}{3}\right)$

(c) (2 points) Fill in the 2D transformation matrix that takes points from F_C to F_A, as given to the right of the above figure.

To form matrix that converts points in F_C to F_A, we express the basis vectors $\mathbf{i}$ *and* $\mathbf{j}$ *of F_C and the translation of F_C's origin in terms of F_A coordinates.*

$$\begin{bmatrix} i_1 & j_1 & t_1 \\ i_2 & j_2 & t_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

To verify, we use the point we computed from part A,

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2/3 \\ 4/3 \\ 1 \end{bmatrix}_C = \begin{bmatrix} 2/3 + 4/3 + 2 \\ 2/3 - 8/3 + 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 6/3 + 2 \\ -6/3 + 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}_A$$

(d) (2 points) Fill in the 2D transformation matrix that takes points from F_A to F_B, as given to the right of the above figure.

We first construct the matrix that converts points from F_B to F_A,

$$\begin{bmatrix} i_1 & j_1 & t_1 \\ i_2 & j_2 & t_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

but now we invert this, giving the matrix that converts points from F_A to F_B as required

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -2 \\ 1 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 2 & -4 \\ 1 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -2 \\ 1 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -2 \\ 0 & 2 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -2 \\ 0 & 1 & 0 & -1/4 & 1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

This leaves the matrix we need on the right side. To verify,

$$\begin{bmatrix} 1/2 & 1/2 & -2 \\ -1/4 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}_A = \begin{bmatrix} 2 + 2 - 2 \\ -1 + 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}_B$$

(e) (2 points) Using the above two matrices, develop a 2D transformation matrix that takes points from F_C to F_B. Test your solution using point P.

Given the matrices from parts C and D, we have M_{CA}, the matrix that converts points from F_C to F_A, and M_{AB}, the matrix that converts points from F_A to F_B. We notice that since $M_{CA}P_C$ gives a point in F_A from a point in F_C, we can use that point to compute the same point in F_B

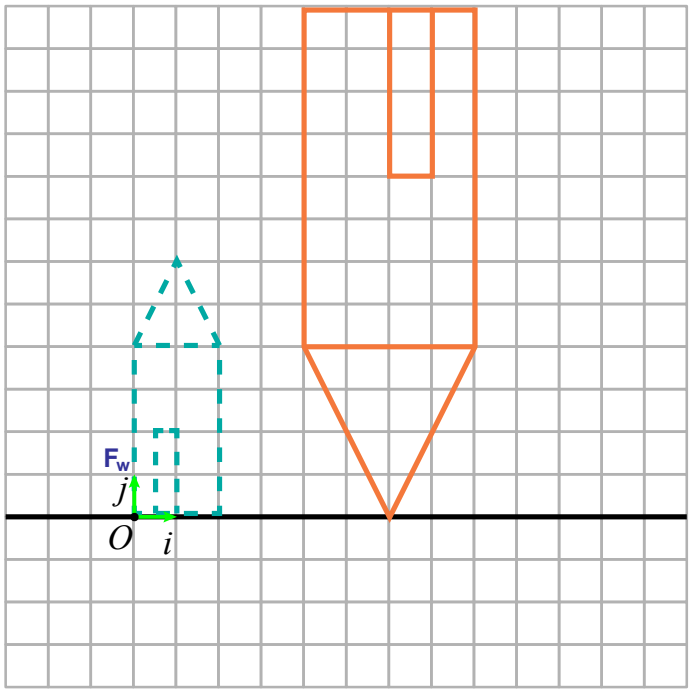
$$\begin{aligned}
P_A &= M_{CA}P_C \\
P_B &= M_{AB}P_A \\
&= M_{AB}\big[M_{CA}P_C\big] \\
\implies P_B &= M_{CB}P_C \\
\implies M_{CB} &= M_{AB}M_{CA} \\
&= \begin{bmatrix} 1/3 & 2/3 & -2 \\ -1/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix} \\
M_{CB} &= \begin{bmatrix} 1 & -1/2 & 2 \\ 0 & -3/4 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

To verify,

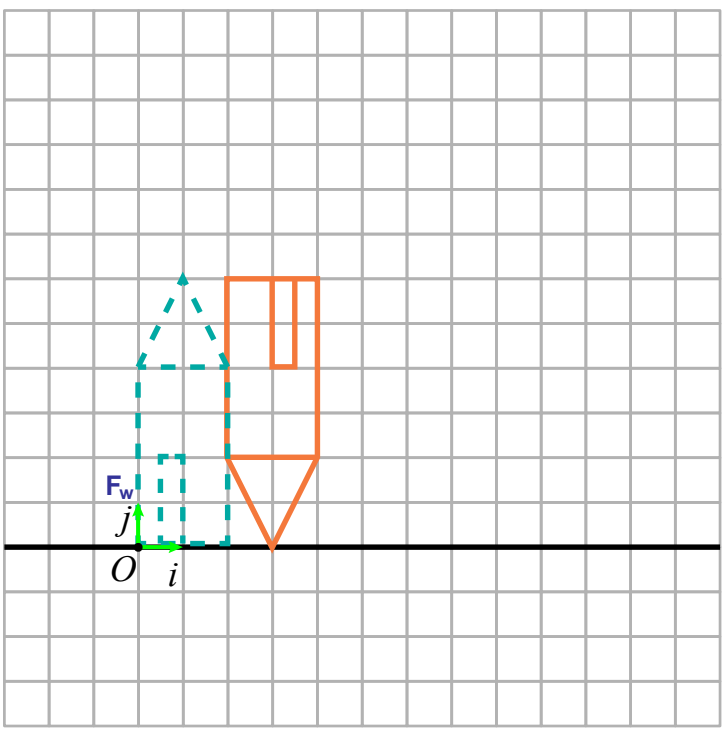
$$\begin{bmatrix} 1 & -1/2 & 2 \\ 0 & -3/4 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2/3 \\ 4/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 - 2/3 + 2 \\ (-3/4)(4/3) + 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

2. Composing Transformations

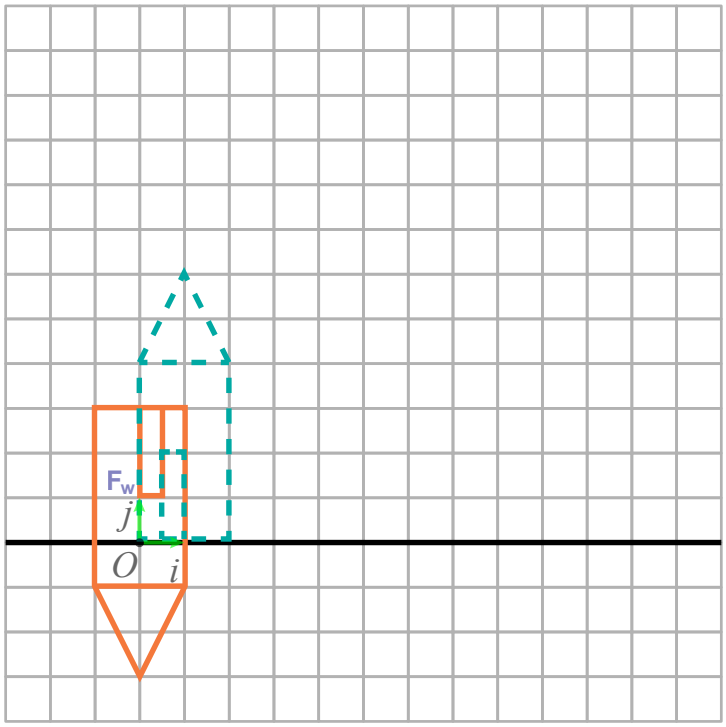
   (a) • We start with the house looking like this



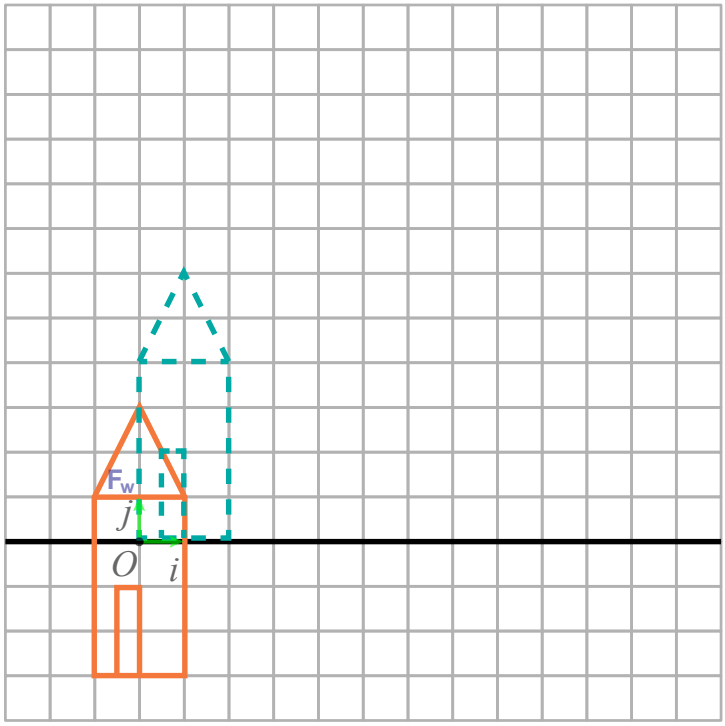
   • Apply a scale factor of $\frac{1}{2}$



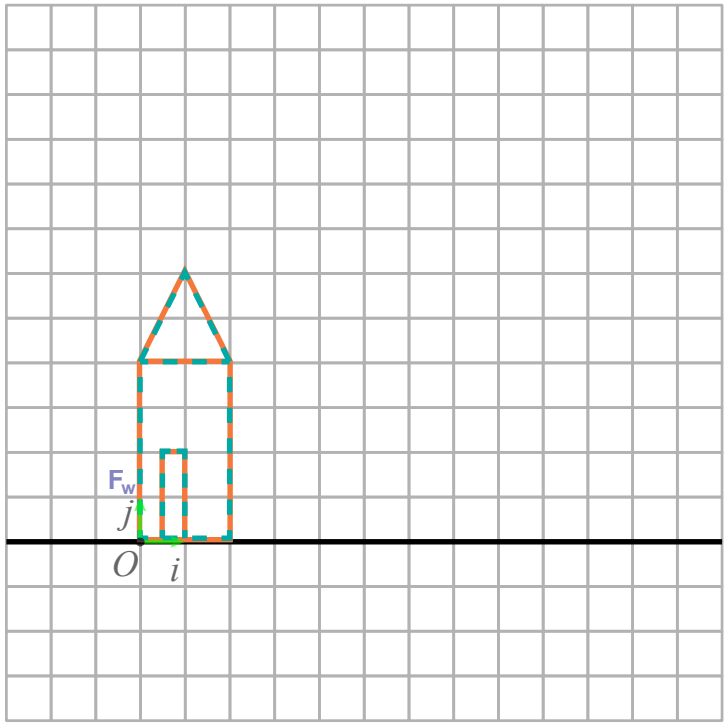
- Translate the house so that the centre is at the origin of the world coordinate frame. That is, we perform a translation of $x - 3, y - 3$



- Rotate the house 180 degrees



- Finally, translate the house to match the dashed outline using a translation of $x + 1, y + 3$



(b) (2 points) Give the resulting 4×4 transformation matrix. Assume that the transformation leaves z to be unaltered.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 0 & 4 \\ 0 & -1/2 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) (2 points) What values would need to be assigned to `theta, a, b, c, i, j, k, x, y, z` in order for the following transformations to yield an identical final transformation? Note, `THREE.Matrix4()` constructs an identity matrix. Also note that here we pretend that * does matrix multiplication. In actual JS code, you'd use multiplyMatrices() function instead.

```
var m = new THREE.Matrix4();
var m1 = new THREE.Matrix4();
var m2 = new THREE.Matrix4();
var m3 = new THREE.Matrix4();
m1.makeRotateAxis(new THREE.Vector3(i=0,j=0,k=1), theta=180);
m2.makeScale(a=0.5,b=0.5,c=1.0);
m3.makeTranslate(x=4,y=6,z=0);
m = m3*m2*m1;
house.geometry.applyMatrix(m);
```

3. Decompose the following complex transformations in homogeneous coordinates into a product of simple transformations (scaling, rotation, translation, shear). Pay attention to the order of transformations.

   (a) (1 point)

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{CCW rotation of 90 degrees} \\ \text{on the Z-axis}}} \cdot \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{CCW rotation of 90 degrees} \\ \text{on the Y-axis}}}$$

   (b) (1 point)

$$\begin{bmatrix} 4 & 0 & 0 & -2 \\ 0 & 0.2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Translation}} \cdot \underbrace{\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Non-uniform scaling}}$$

   (c) (2 points) What are the inverses of the matrices of parts (a) and (b) above?
   Hint: if $M = AB$, then $M^{-1} = B^{-1}A^{-1}$

   Part A: *It is easy to think of the inverse of some rotation of θ degrees as the rotation about the same axis of $-\theta$ degrees. Then using the hint, we find*

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   Further note: it is useful to recall that sin *is an odd function and* cos *is an even function. Thus, the inversions become a flip of the sign for entries with* sin*, but the entries with* cos *do not change signs.*

   Part B:

$$\begin{bmatrix} 4 & 0 & 0 & -2 \\ 0 & 0.2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 & 0 & 1/2 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d) (2 points) Give the sequence of THREE.js transformations that would produce the same transformation matrix as in part (a) of this question.

```
var m = new THREE.Matrix4();
var m1 = new THREE.Matrix4();
var m2 = new THREE.Matrix4();
m1.makeRotateAxis(new THREE.Vector3(0,0,1), 90);
m2.makeRotateAxis(new THREE.Vector3(0,1,0), 90);
m = m1*m2;
```

4. Rotation Matrices

(a) (2 points) The columns of a rotation matrix have unit magnitude and they should all be orthogonal to each other, i.e., have a zero dot product. Show that the inverse of a rotation matrix is given by its transpose.

Let r_1, r_2, r_3 be the column vectors in some rotation matrix R. Suppose that the column vectors are orthogonal to each other so that $r_1 \cdot r_2 = r_2 \cdot r_3 = r_3 \cdot r_1 = 0$
We want to show that $R^T = R^{-1}$

$$R^T = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix} \implies R^T R = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix} \cdot \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$$
$$= \begin{bmatrix} r_1 \cdot r_1 & r_1 \cdot r_2 & r_1 \cdot r_3 \\ r_2 \cdot r_2 & r_2 \cdot r_2 & r_2 \cdot r_3 \\ r_3 \cdot r_3 & r_3 \cdot r_2 & r_3 \cdot r_3 \end{bmatrix}$$

It is simple to use the formula for the dot product

$$a \cdot b = \|a\| \cdot \|b\| \cos\theta$$

to see that the diagonal entries $r_1 \cdot r_1 = r_2 \cdot r_2 = r_3 \cdot r_3 = 1$ since the column vectors are unit and $\theta = 0$ given the dot product is operating over identical vectors

$$r_i \cdot r_i = \cos\theta = 1, \ i = \{1, 2, 3\}$$

Now recall that the columns are orthogonal to each other, so we have

$$\left(R^T R\right)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

so we get $R^T R = I$. By definition of the inverse, the multiplication of a matrix with its inverse gives the identity. Thus we have shown that $R^T = R^{-1}$.

(b) (4 points) In order for a rotation matrix to represent a rigid body rotation, there is one more constraint that it should satisfy, in addition to those listed above. Which of the following 4×4 matrices represent a valid rotation matrix? Why or why not? What is the extra constraint that rotation matrices should satisfy?

'Proper rotations' are members of the special orthogonal group. These preserve length and orientation (as opposed to reflections or compositions of reflections and rotations which invert orientation) and so have determinant $+1$. Reflections on the other hand have determinant -1.

- *We see that $\det(A) = 1$, but the columns of A are not orthogonal and they are not unit, so A is not a rotation matrix.*
- *But $\det(B) = -1$, so B is not a rotation. We can say B is a reflection however.*
- *Since $\det(C) = 0$, C is not a rotation matrix.*
- *Since $\det(D) = 1$, we can say D is a rotation matrix.*