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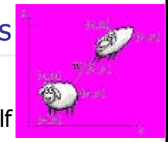
Chapter 3

Transformations

Geometric Transformations



- Transformation τ is *one-to-one* and *onto* mapping of nD to itself
- *Affine* transformation – $\tau(V) = AV+b$
  - A – matrix
  - b – scalar
- In CG *Geometric Transformations* are affine transformations with geometric meaning
- Note: transformation has meaning only for vectors $\Rightarrow$ for point P $\tau(P)$ is transformation of vector from (0,0) to P

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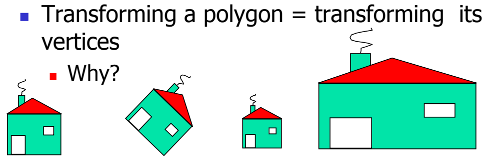
Transformations

- Transforming an object = transforming all its points
- Transforming a polygon = transforming its vertices
  - Why?



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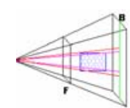
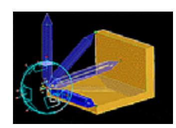
Applications

- Viewing
- Modeling
- Articulation





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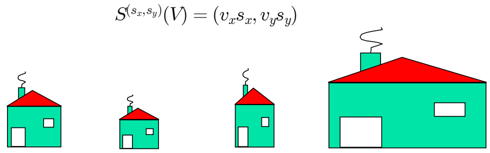
Scaling

- $V = (v_x, v_y)$ – vector in XY plane
- *Scaling* operator S with parameters (s_x, s_y):

$$S^{(s_x,s_y)}(V) = (v_x s_x, v_y s_y)$$



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Scaling

- Matrix form:

$$S^{(s_x,s_y)}(V) = (v_x, v_y)\begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} = (v_x s_x, v_y s_y)$$

- Independent in x and y

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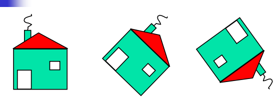
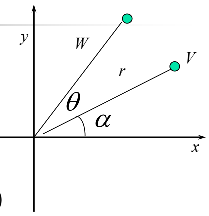
Rotation





- Polar form:

$V = (v_x, v_y) = (r\cos\alpha, r\sin\alpha)$

- Rotating V counterclockwise by θ to W:

$$W = (w_x, w_y) = (r\cos(\alpha+\theta), r\sin(\alpha+\theta)) = (r\cos\alpha\cos\theta - r\sin\alpha\sin\theta,\ r\cos\alpha\sin\theta + r\sin\alpha\cos\theta)$$

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Rotation

- Matrix form:

$$W = (r\cos\alpha, r\sin\alpha)\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} = V\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$$

- *Rotation* operator R with parameter θ at the origin:

$$R^\theta = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$$

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Rotation Properties

- R^θ is orthogonal

$$(R^\theta)^{-1} = (R^\theta)^T$$

- $R^{-\theta}$ - rotation by $-\theta$ is

$$R^{-\theta}(V) = (v_x, v_y)\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} = (R^\theta)^{-1}$$

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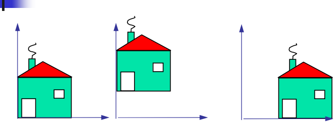
Translation



- Translation operator T with parameters (t_x, t_y):

$$T^{(t_x,t_y)}(V) = (v_x + t_x, v_y + t_y)$$

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Translation - Homogeneous Coordinates

- To represent T in matrix form – introduce homogeneous coordinates:

$$V^h = (v^h_x, v^h_y, v^h_w) = (v_x, v_y, 1)$$

- Conversion (projection) from homogeneous space to Euclidean:

$$V = (v_x, v_y) = \left(\frac{v^h_x}{v^h_w}, \frac{v^h_y}{v^h_w}\right)$$

- In homogeneous coordinates:

(2,2,1)=(4,4,2)=(1,1,0.5)

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Translation

- Using homogeneous coordinates, translation operator may be expressed as:

$$T^{(t_x,t_y)}(V^h) = (v_x, v_y, 1)\begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1\end{bmatrix} = (v_x + t_x, v_y + t_y, 1)$$

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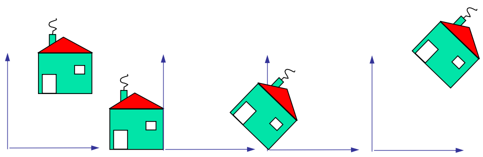
Transformation Composition

- What operation rotates XY by θ around $P = (p_x, p_y)$?
- Answer:
  - Translate P to origin
  - Rotate around origin by θ
  - Translate back

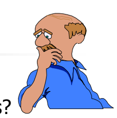


Transformation Composition

$$T^{(-p_x,-p_y)} R^{\theta} T^{(p_x,p_y)} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p_x & -p_y & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_x & p_y & 1 \end{bmatrix}$$

Transformations Quiz



- What do these transformations do?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

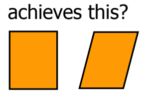
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

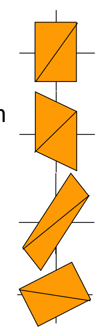
- And these homogeneous ones?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

- How to mirror through arbitrary line in XY?
- What transformation achieves this?



Rotate by Shear



- Shear

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

- Rotation by $0 \le \theta \le \frac{\pi}{2} \equiv$ composition of 3 shears

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+xy & x \\ y & 1 \end{bmatrix} \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+xy & z+xyz+x \\ y & yz+1 \end{bmatrix}.$$

Rotate by Shear (cont.)

- Solve for x, y, z :

$$\cos\theta = 1 + xy = yz + 1, \sin\theta = -y = z + xyz + x$$

- Result

$$y = -\sin\theta, x = z = -tan\tfrac{\theta}{2}$$

Rotate by Shear (cont.)

- What does composition of following two scaled shears do?

$$\begin{bmatrix} 1 & \sin\theta \\ 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \sec\theta & 0 \\ -\tan\theta & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

- Why use it?
- Is it convenient?
- What happens when $\theta \rightarrow \frac{\pi}{2}$?

Rotation Approximation

- For small angles can approximate:

$$x' = x\cos\theta - y\sin\theta \simeq x - y\theta$$
$$y' = x\sin\theta + y\cos\theta \simeq x\theta + y$$

  - $\cos\theta \to 1, \sin\theta \to \theta$ as $\theta \to 0$
  - (Taylor expansion of sin/cos)
- Example:

25° 10° 2° 0.5°

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Rotation Approximation

- App. rotation matrix

$$AR^\theta = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}$$

- Matrix has to be orthogonal & normalized
- AR^θ is orthogonal
- AR^θ not normalized
- Normalize

$$AR^\theta = \begin{pmatrix} \frac{1}{c} & \frac{\theta}{c} \\ \frac{-\theta}{c} & \frac{1}{c} \end{pmatrix}, c = \sqrt{(1+\theta^2)}$$

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3D Transformations

- All 2D transformations extend to 3D
- In homogeneous coordinates:

Scaling

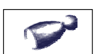
$$S^{(s_x,s_y,s_z)} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation

$$T^{(t_x,t_y,t_z)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

Rotation around the z axis

$$R_z^\theta = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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3D Transformations

- Questions:
  - Is $S_1S_2 = S_2S_1$?
  - Is $T_1T_2 = T_2T_1$?
  - Is $R_1R_2 = R_2R_1$?
  - Is $S_1R_2 = R_2S_1$?
  - .....

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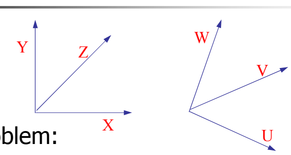
Example: Arbitrary Rotation



- Problem:
  - Given two orthonormal coordinate systems XYZ and UVW
  - Find transformation from one to the other
- Answer:
  - Transformation matrix R whose rows are U,V,W:

$$R = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}$$

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Arbitrary Rotation

- Proof:

$$R(X) = (1,0,0)\begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} = (u_x, u_y, u_z) = U$$

- Similarly $R(Y) = V$ & $R(Z) = W$

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Arbitrary Rotation (cont.)

- Inverse (=transpose) transformation R^{-1} provides mapping from *UVW* to *XYZ*
- E.g.

$$R^{-1}(U) = (u_x, u_y, u_z)\begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}$$
$$= (u_x^2 + u_y^2 + u_z^2, 0, 0)$$
$$= (1,0,0)$$
$$= X$$

- Comment: Used for mapping between XY and arbitrary plane

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Transformation of Objects

- Lemma: Affine image of line segment is line segment between image of its end points
- Proof:
  - Given affine transformation A & line segment $S(t) = P_1 t + P_2(1-t)$

$$A(S(t)) = A(P_1 t + P_2(1-t))$$
$$= A(P_1 t) + A(P_2(1-t))$$
$$= tA(P_1) + (1-t)A(P_2)$$

- Conclusion: polyhedron(polygon) is affinely transformed by transforming its vertices (Why?)

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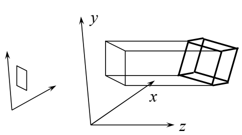
Viewing Transformations



- Question: How to view (draw) 3D object on 2D screen?
- Answer:
  - Project transformed object along *Z* axis onto *XY* plane - and from there to screen (space)
  - Canonical projection:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- In practice "ignore" *z* axis – use *x* and *y* coordinates for screen coordinates

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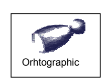
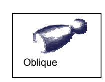
Parallel Projection

- Projectors are all parallel
  - Orthographic: Projectors perpendicular to projection plane
  - Oblique: Projectors not necessarily perpendicular to projection plane





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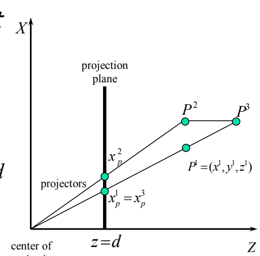
Perspective Projection

- Viewing is from *point at finite distance*
- Without loss of generality:
  - Viewpoint at origin
  - Viewing plane is $z=d$
- Given P=(x,y,z) triangle similarity gives:



$$\frac{x}{z} = \frac{x_p}{d} \text{ and } \frac{y}{z} = \frac{y_p}{d} \Rightarrow x_p = \frac{x}{z/d} \text{ and } y_p = \frac{y}{z/d}$$

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Perspective Projection (cont'd)

- In matrix notation with homogeneous coordinates:

$$P(x,y,z,1) = (x,y,z,1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/d \\ 0 & 0 & 0 & 0 \end{bmatrix} = (x,y,z,\frac{z}{d}).$$

- In Euclidean coordinates:

$$\left(\frac{x}{z/d}, \frac{y}{z/d}, \frac{z}{z/d}\right) = \left(\frac{x}{z/d}, \frac{y}{z/d}, d\right) = (x_p, y_p, d).$$

- *P* singular: $det(P)=0$

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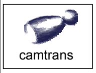
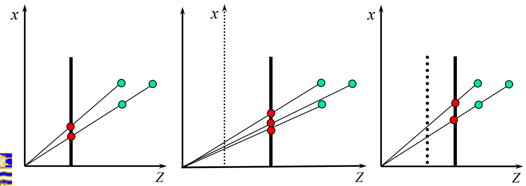
Perspective Projection (cont'd)

- What is (if any) is the difference between:
  - Moving projection plane
  - Moving viewpoint (center of projection)?





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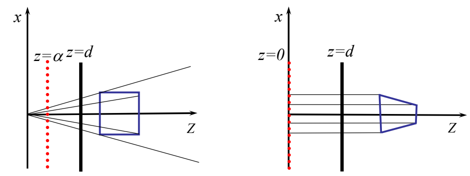
Perspective Warp

- P not invertible – can't get depth (order) back
- Idea:
  - Warp viewing frustum (world)
  - Then use parallel projection



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Perspective Warp

- Matrix formulation

$$(x,y,z,1)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{d}{d-\alpha}&\frac{1}{d}\\0&0&\frac{-\alpha d}{d-\alpha}&0\end{bmatrix}=\left(x,y,\frac{(z-\alpha)d}{d-\alpha},\frac{z}{d}\right)$$

$$(x_p,y_p,z_p)=\left(\frac{x}{z/d},\frac{y}{z/d},\frac{d^2}{d-\alpha}\left(1-\frac{\alpha}{z}\right)\right)$$

- Preserves relative depth (third coordinate)
- What does $\alpha=0$ mean?

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Perspective Warp

- Warp modifies planes

- $Ax+By+Cz+D=0$ warped to:

$$A\alpha dx_p+B\alpha dy_p+D(\alpha-d)z_p+C\alpha d^2+Dd^2=0$$

- What does warp do to other objects? Spheres?

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Another Transformations Quiz

- What does each transformation preserve?

	lines	parallel lines	distance	angles	normals	convexity	conics
scaling							
rotation							
translation							
shear							
perspective							

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