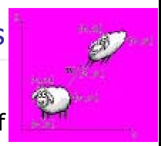
Chapter 3

Transformations

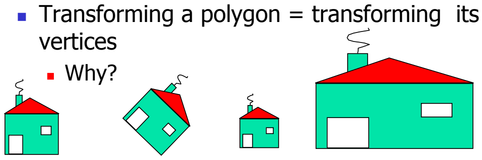

## Geometric Transformations



- Transformation $\tau$ is *one-to-one* and *onto* mapping of $nD$ to itself
- *Affine* transformation – $\tau(V) = AV+b$
  - $A$ – matrix
  - $b$ – scalar
- In CG *Geometric Transformations* are affine transformations with geometric meaning
- Note: transformation has meaning only for vectors $\Rightarrow$ for point $P$ $\tau(P)$ is transformation of vector from $(0,0)$ to $P$

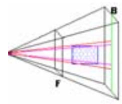
## Transformations

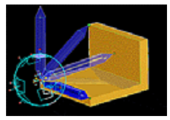
- Transforming an object = transforming all its points
- Transforming a polygon = transforming its vertices
  - Why?



## Applications

- Viewing
- Modeling
- Articulation





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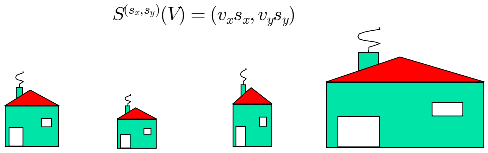
## Scaling

- V = $(v_x, v_y)$ – vector in XY plane
- *Scaling* operator $S$ with parameters $(s_x, s_y)$:

$$S^{(s_x,s_y)}(V) = (v_x s_x, v_y s_y)$$



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## Scaling

- Matrix form:

$$S^{(s_x,s_y)}(V) = (v_x, v_y)\begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} = (v_x s_x, v_y s_y)$$

- Independent in $x$ and $y$

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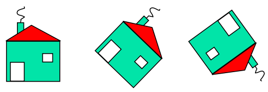
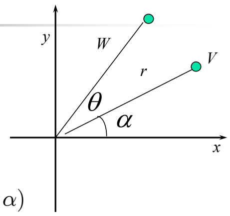
## Rotation






- Polar form:

$$V=(v_x,v_y)=(r\cos\alpha, r\sin\alpha)$$

- Rotating V counterclockwise by $\theta$ to W:

$$\begin{aligned}W&=(w_x,w_y)\\&=(r\cos(\alpha+\theta), r\sin(\alpha+\theta))\\&=(r\cos\alpha\cos\theta-r\sin\alpha\sin\theta,\\&\quad r\cos\alpha\sin\theta+r\sin\alpha\cos\theta)\end{aligned}$$

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## Rotation

- Matrix form:

$$W=(r\cos\alpha, r\sin\alpha)\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$
$$=V\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

- *Rotation* operator $R$ with parameter $\theta$ at the origin:

$$R^\theta=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

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## Rotation Properties

- $R^\theta$ is orthogonal

$$\left(R^\theta\right)^{-1}=\left(R^\theta\right)^T$$

- $R^{-\theta}$ - rotation by $-\theta$ is

$$R^{-\theta}(V)=(v_x,v_y)\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\left(R^\theta\right)^{-1}$$

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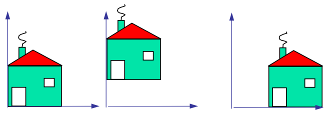
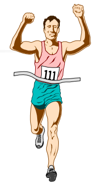
## Translation





- Translation operator $T$ with parameters $(t_x, t_y)$:

$$T^{(t_x,t_y)}(V) = (v_x + t_x, v_y + t_y)$$

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## Translation - Homogeneous Coordinates

- To represent $T$ in matrix form – introduce homogeneous coordinates:

$$V^h = (v_x^h, v_y^h, v_w^h) = (v_x, v_y, 1)$$

- Conversion (projection) from homogeneous space to Euclidean:

$$V = (v_x, v_y) = \left(\frac{v_x^h}{v_w^h}, \frac{v_y^h}{v_w^h}\right)$$

- In homogeneous coordinates:

*(2,2,1)=(4,4,2)=(1,1,0.5)*

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## Translation

- Using homogeneous coordinates, translation operator may be expressed as:

$$T^{(t_x,t_y)}(V^h) = (v_x, v_y, 1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix} = (v_x + t_x, v_y + t_y, 1)$$

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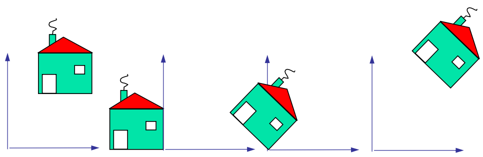
## Transformation Composition

- What operation rotates XY by $\theta$ around $P = (p_x, p_y)$ ?
- Answer:
  - Translate $P$ to origin
  - Rotate around origin by $\theta$
  - Translate back



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## Transformation Composition

$$T^{(-p_x,-p_y)} R^{\theta} T^{(p_x,p_y)} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p_x & -p_y & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_x & p_y & 1 \end{bmatrix}$$

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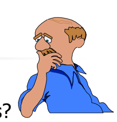
## Transformations Quiz



- What do these transformations do?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

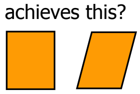
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

- And these homogeneous ones?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

- How to mirror through arbitrary line in XY?
- What transformation achieves this?



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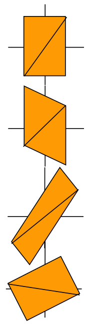
## Rotate by Shear

- Shear

$$\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix},\begin{bmatrix}1 & 0\\ a & 1\end{bmatrix}$$

- Rotation by $0 \le \theta \le \frac{\pi}{2} \equiv$ composition of 3 shears

$$\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ y & 1\end{bmatrix}\begin{bmatrix}1 & z\\ 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}1+xy & x\\ y & 1\end{bmatrix}\begin{bmatrix}1 & z\\ 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}1+xy & z+xyz+x\\ y & yz+1\end{bmatrix}.$$



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## Rotate by Shear (cont.)

- Solve for $x, y, z$ :

$$\cos\theta = 1 + xy = yz + 1, \sin\theta = -y = z + xyz + x$$

- Result

$$y = -\sin\theta, x = z = -tan\tfrac{\theta}{2}$$

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## Rotate by Shear (cont.)

- What does composition of following two scaled shears do?

$$\begin{bmatrix}1 & \sin\theta\\ 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\sec\theta & 0\\ -\tan\theta & 1\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

- Why use it?
- Is it convenient?
- What happens when $\theta \rightarrow \frac{\pi}{2}$ ?

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## Rotation Approximation

- For small angles can approximate:

$$x' = x\cos\theta - y\sin\theta \simeq x - y\theta$$
$$y' = x\sin\theta + y\cos\theta \simeq x\theta + y$$

  - $\cos\theta \to 1, \sin\theta \to \theta$ as $\theta \to 0$
  - (Taylor expansion of sin/cos)
- Example:


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## Rotation Approximation

- App. rotation matrix

$$AR^{\theta} = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}$$

- Matrix has to be orthogonal & normalized
- $AR^{\theta}$ is orthogonal
- $AR^{\theta}$ not normalized
- Normalize

$$AR^{\theta} = \begin{pmatrix} \frac{1}{c} & \frac{\theta}{c} \\ \frac{-\theta}{c} & \frac{1}{c} \end{pmatrix}, c = \sqrt{(1+\theta^2)}$$

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## 3D Transformations

- All 2D transformations extend to 3D
- In homogeneous coordinates:

Scaling

$$S^{(s_x,s_y,s_z)} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation

$$T^{(t_x,t_y,t_z)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

Rotation around the z axis

$$R_z^{\theta} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$


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## 3D Transformations

- Questions:
  - Is $S_1S_2 = S_2S_1$?
  - Is $T_1T_2 = T_2T_1$?
  - Is $R_1R_2 = R_2R_1$?
  - Is $S_1R_2 = R_2S_1$?
  - .....

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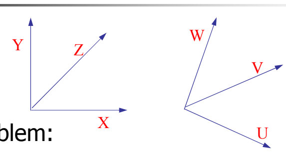
## Example: Arbitrary Rotation




- Problem:
  - Given two orthonormal coordinate systems $XYZ$ and $UVW$
  - Find transformation from one to the other
- Answer:
  - Transformation matrix R whose rows are $U, V, W$:

$$R = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}$$

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## Arbitrary Rotation

- Proof:

$$R(X) = (1,0,0)\begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}$$
$$= (u_x, u_y, u_z)$$
$$= U$$

- Similarly $R(Y) = V$ & $R(Z) = W$

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## Arbitrary Rotation (cont.)

- Inverse (=transpose) transformation $R^{-1}$ provides mapping from $UVW$ to $XYZ$
- E.g.

$$R^{-1}(U)=(u_x,u_y,u_z)\begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}$$
$$=(u_x^2+u_y^2+u_z^2,0,0)$$
$$=(1,0,0)$$
$$=X$$

- Comment: Used for mapping between XY and arbitrary plane

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## Transformation of Objects

- Lemma: Affine image of line segment is line segment between image of its end points
- Proof:
  - Given affine transformation A & line segment $S(t)=P_1t+P_2(1-t)$

$$A(S(t))=A(P_1t+P_2(1-t))$$
$$=A(P_1t)+A(P_2(1-t))$$
$$=tA(P_1)+(1-t)A(P_2)$$

- Conclusion: polyhedron(polygon) is affinely transformed by transforming its vertices (Why?)

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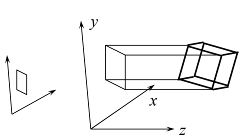
## Viewing Transformations




- Question: How to view (draw) 3D object on 2D screen?
- Answer:
  - Project transformed object along $Z$ axis onto $XY$ plane - and from there to screen (space)
  - Canonical projection:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- In practice "ignore" *z* axis – use *x* and *y* coordinates for screen coordinates

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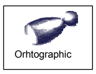
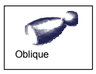
## Parallel Projection

- Projectors are all parallel
  - Orthographic: Projectors perpendicular to projection plane
  - Oblique: Projectors not necessarily perpendicular to projection plane

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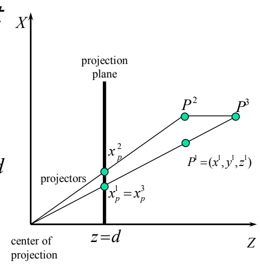
## Perspective Projection

- Viewing is from *point at finite distance*
- Without loss of generality:
  - Viewpoint at origin
  - Viewing plane is $z=d$
- Given P=(x,y,z) triangle similarity gives:




$$\frac{x}{z} = \frac{x_p}{d} \text{ and } \frac{y}{z} = \frac{y_p}{d} \Rightarrow x_p = \frac{x}{z/d} \text{ and } y_p = \frac{y}{z/d}$$

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## Perspective Projection (cont'd)

- In matrix notation with homogeneous coordinates:

$$P(x,y,z,1)=(x,y,z,1)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1/d\\0&0&0&0\end{bmatrix}=(x,y,z,\frac{z}{d}).$$

- In Euclidean coordinates:

$$\left(\frac{x}{z/d},\frac{y}{z/d},\frac{z}{z/d}\right)=\left(\frac{x}{z/d},\frac{y}{z/d},d\right)=(x_p,y_p,d).$$


- $P$ singular: $det(P)=0$

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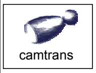
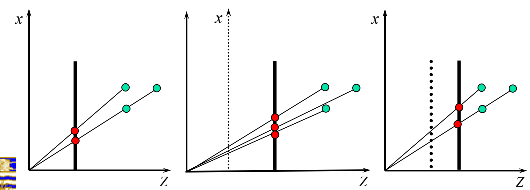
## Perspective Projection (cont'd)

- What is (if any) is the difference between:
  - Moving projection plane
  - Moving viewpoint (center of projection)?

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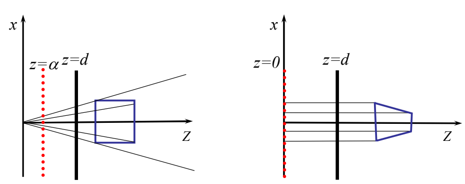
## Perspective Warp

- $P$ not invertible – can't get depth (order) back
- Idea:
  - Warp viewing frustum (world)
  - Then use parallel projection

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## Perspective Warp

- Matrix formulation

$$(x,y,z,1)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \frac{d}{d-\alpha} & \frac{1}{d}\\ 0 & 0 & \frac{-\alpha d}{d-\alpha} & 0\end{bmatrix} = \left(x,y,\frac{(z-\alpha)d}{d-\alpha},\frac{z}{d}\right)$$

$$(x_p,y_p,z_p) = \left(\frac{x}{z/d},\frac{y}{z/d},\frac{d^2}{d-\alpha}\left(1-\frac{\alpha}{z}\right)\right)$$

- Preserves relative depth (third coordinate)
- What does $\alpha = 0$ mean?

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## Perspective Warp

- Warp modifies planes
- $Ax + By + Cz + D = 0$ warped to:

$$A\alpha d x_p + B\alpha d y_p + D(\alpha - d) z_p + C\alpha d^2 + Dd^2 = 0$$

- What does warp do to other objects? Spheres?

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## Another Transformations Quiz

- What does each transformation preserve?

| | lines | parallel lines | distance | angles | normals | convexity | conics |
|---|---|---|---|---|---|---|---|
| scaling | | | | | | | |
| rotation | | | | | | | |
| translation | | | | | | | |
| shear | | | | | | | |
| perspective | | | | | | | |

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