Chapter 6

Geometric Modeling
Part I: Terminology & Splines

An Example
Operations

Acquisition/Generation

Transmission/Storage

Editing

Geometry

- Mathematical models of real world objects shape
- Categories:
  - Boundary representations
    - Freeform
    - Meshes & subdivision
  - Volumetric representations
    - Primitive based
    - Voxels
Volumetric - Volxelization

- Voxel based
  - Space subdivided into equal size boxes – each has in/out flag
  - Common in imaging applications (CT,MRI,Ultrasound)
- Extension – Octree (recursive construction)
- To draw use *iso-surfaces* – boundary between voxels with different flag

Volumetric - Primitives

- Use set of volumetric primitives
  - Box, sphere, cylinder, cone, etc...
- For complex objects use boolean operations
  - Union
  - Intersection
  - Subtraction
Freeform Representation

- Explicit form: $z = z(x, y)$
- Implicit form: $f(x, y, z) = 0$
- Parametric form: $[x(u, v), y(u, v), z(u, v)]$
- Example – origin centered sphere of radius $R$:

  **Explicit:**
  
  $$z = \sqrt{R^2 - x^2 - y^2} \cup z = -\sqrt{R^2 - x^2 - y^2}$$

  **Implicit:**
  
  $$x^2 + y^2 + z^2 - R^2 = 0$$

  **Parametric:**
  
  $$(x, y, z) = (R \cos \theta \cos \psi, R \sin \theta \cos \psi, R \sin \psi), \theta \in [0, 2\pi], \psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

Splines – Free Form Curves

- Description = basis functions + coefficients
- Geometric meaning of coefficients (base)
  - Approximate/interpolate set of positions, derivatives, etc..

- Usually parametric
Polynomial Bases

- Monomial basis
  - Geometrically meaningless \{1, t, t^2, t^3, \ldots\}
- Other bases
  - Bezier
  - Hermitte
  - Lagrange, B-Spline, ...
- Number of coefficients = polynomial rank
- Geometric meaning
  - coefficients - positions/derivatives, etc...
- Advantage
  - easy to analyze, derivatives remain polynomial

Lagrange

- Interpolates *control points*
  - \(P_0, \ldots, P_n\)
- Base function \(P_i(t)\) per control-point \(P_i\)
  - \(P_i(t) = 1\) at \(i/n\)
  - \(P_i(t) = 0\) at \(j/n, j \neq i\)
- Each base function is a polynomial of degree \(n\)
  \[P_i(t) = a_i^0 + a_i^1 t + a_i^2 t^2 + \cdots + a_i^n t^n\]
- Uniqueness
  - \(n\) equations in \(n\) unknowns \(\{a_i^j\}\)
Analytic Continuity

- $C_1(t)$ & $C_2(t)$, $t \in [0,1]$ - parametric curves
- Level of continuity at $C_1(1)$ and $C_2(0)$ is:
  - $C^1: C_1(1) \neq C_2(0)$ (discontinuous)
  - $C^0: C_1(1) = C_2(0)$ (positional continuity)
  - $C^k, k > 0$: continuous up to $k$-th derivative

Continuity of single curve defined similarly
- for polynomial bases $C^\infty$

Geometric Continuity

- Analytic continuity - too strong a requirement
- Geometric continuity – common curve is geometrically smooth (per given level $k$)
  - $G^k, k \leq 0$ : Same as $C^k$
  - $G^k k = 1: C'_1 (1) = \alpha C'_2(0)$ $\alpha > 0$
  - $G^k k \geq 0$ : In arc-length reparameterization of $C_1(t)$ & $C_2(t)$, the two are $C^k$
Geometric Continuity

- E.g.
  \[ C_1(t) = [\cos(t), \sin(t)] \quad t \in [-0.5\pi, 0] \]
  \[ C_2(t) = [\cos(t), \sin(t)] \quad t \in [0, 0.5\pi] \]
  \[ C_3(t) = [\cos(2t), \sin(2t)] \quad t \in [0, 0.25\pi] \]

- \( C_1(t) \) & \( C_2(t) \) are \( C^k \) (& \( G^k \)) continuous
- \( C_2(t) \) & \( C_3(t) \), are \( G^k \) continuous (not \( C^k \))

Hermite Cubic Basis

- Geometrically-oriented basis for cubic polynomials
  - 2 positions + 2 tangents
  - Has to satisfy

\[ h_{i,j}(t): i, j = 0, 1, \quad t \in [0, 1] \]

<table>
<thead>
<tr>
<th>curve</th>
<th>( h(0) )</th>
<th>( h(1) )</th>
<th>( h'(0) )</th>
<th>( h'(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_{00}(t) )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( h_{01}(t) )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( h_{10}(t) )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( h_{11}(t) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Hermite Cubic Basis (cont’d)

- Four cubics which satisfy the conditions:
  \[ h_{00}(t) = t^2(2t - 3) + 1 \]
  \[ h_{01}(t) = -t^2(2t - 3) \]
  \[ h_{10}(t) = t(t - 1)^2 \]
  \[ h_{11}(t) = t^2(t - 1) \]

- Obtain - solve 4 linear equations in 4 unknowns for each basis function.

Hermite Cubic Basis (cont’d)

- Let \( C(t) \) be a cubic polynomial defined as the linear combination:
  \[ C(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t) \]

- What are \( C(0), C(1), C'(0), C'(1) \)?
  - \( C(0) = P_0, C(1) = P_1, C'(0) = T_0, C'(1) = T_1 \)

  - To generate a curve through \( P_0 \) & \( P_1 \) with slopes \( T_0 \) & \( T_1 \) use
  \[ C(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t) \]
Natural Cubic Splines

- Standard spline input – set of points \( \{P_i\}_{i=0}^n \)
- No derivatives
- Interpolate by \( n \) cubic segments:
  - Derive \( \{T_i\}_{i=0}^n \) from continuity constraints
  - Solve \( 4n \) equations

Interpolation (2n equations):
\[
C_i(0) = P_{i-1}, \quad C_i(1) = P_i, \quad i = 1,\ldots,n
\]

\( C^0 \) continuity constraints (\( n-1 \) equations):
\[
C_i'(1) = C_{i+1}'(0), \quad i = 1,\ldots,n-1
\]

\( C^1 \) continuity constraints (\( n-1 \) equations):
\[
C_i''(1) = C_{i+1}''(0), \quad i = 1,\ldots,n-1
\]

Need another 2 equations to reach \( 4n \)

Options

- Natural end conditions: \( C_i''(0) = 0, C_n''(1) = 0 \)
- Prescribed end conditions (derivative available): \( C_i'(0) = T_0, C_n'(1) = T_n \)
B-Splines

- Idea: Generate basis where functions are continuous across domains

- Control point controls set of basis functions (to preserve continuity)

- Alternative view: continuous basis functions defined on several domains

Uniform Cubic B-Spline Curves

- Definition

\[ C(t) = \sum_{i=0}^{n-1} P_i N_i^3(t) \quad t \in [3, n] \]

\[ N_i^3(t) = \begin{cases} 
  r^3 / 6 & r = t - i \\ 
  (-3r^3 + 3r^2 + 3r + 1) / 6 & r = t - i - 1 \\ 
  (3r^3 - 6r^2 + 4r + 1) / 6 & r = t - i - 2 \\ 
  (1 - r)^3 / 6 & r = t - i - 3 \\ 
  0 & \text{otherwise} 
\]
Uniform Cubic B-Spline Curves

- For any $t \in [3, n]$ \( \sum_{i=j-3}^{j} N_i^3(t) = 1 \)

- For any $t \in [j, j+1]$ only 4 basis functions are non zero

- Any point on cubic B-Spline is affine combination of at most 4 control points

\[ N_j^3(t) = \sum_{i=j-3}^{j} N_i^3(t) \]

Boundary Conditions for B-Splines

- B-Splines do not interpolate any control points
  - in particular end points

- Way to force endpoint interpolation:
  - Let \( P_0 = P_1 = P_2 \) and same for other end

- Question:
  - What is the shape of the curve at endpoints if this method is used?
B-Spline Curve Properties

- For $n$ control points, $C(t)$ is a piecewise polynomial of degree 3, defined for

  $$C(t) = \bigcup_{i=0}^{n-3} CH(P_i, ..., P_{i+3}) \quad t \in [3,n]$$

- $C(t)$ is affine invariant

Questions:
- What is $C(i)$ ?
- What is $C'(i)$ ?
- What is the continuity of $C(t)$ ?

NURBs

- B-Spline

  $$C(t) = \sum_{i=0}^{n-1} P_i N_i^3(t) \quad t \in [3,n]$$

  $$N_i^3(t) = \begin{cases} 
  r^3 / 6 & r = t - i \quad t \in [i,i+1] \\
  (-3r^3 + 3r^2 + 3r + 1) / 6 & r = t - i - 1 \quad t \in [i+1,i+2] \\
  (3r^3 - 6r^2 + 4) / 6 & r = t - i - 2 \quad t \in [i+2,i+3] \\
  (1-r)^3 / 6 & r = t - i - 3 \quad t \in [i+3,i+4] 
  \end{cases}$$

- Non-Uniform – different interval lengths (knots)
- Rational – rational basis functions

  $$C(t) = \frac{\sum_{i=0}^{n-1} w_i P_i N_i^3(t)}{\sum_{i=0}^{n-1} w_i N_i^3(t)} \quad t \in [3,n]$$
From Curves to Surfaces

- Curve is expressed as inner product of \( P_i \) coefficients and basis functions
  \[ C(u) = \sum_{i=0}^{n} P_i B_i(u) \]
- To extend curves to surfaces - treat surface as a curve of curves
- Assume \( P_i \) is not constant, but a function of second parameter \( v \):
  \[ P_i(v) = \sum_{j=0}^{m} Q_{ij} B_j(v) \]
  \[ C(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} Q_{ij} B_j(v) B_i(u) \]

Bilinear Patches

- Bilinear interpolation of 4 3D points - 2D analog of 1D linear interpolation between 2 points in the plane
- Given \( P_{00}, P_{01}, P_{10}, P_{11} \) associated parametric bilinear surface for \( u, v \in [0,1] \) is:
  \[ P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11} \]
- Questions:
  - What does an isoparametric curve of a bilinear patch look like?
  - When is a bilinear patch planar?