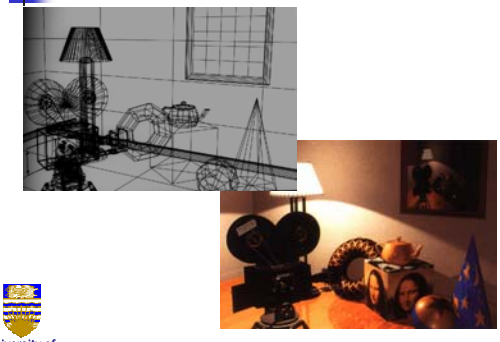
Chapter 6

Geometric Modeling
Part I: Terminology & Splines

University of British Columbia

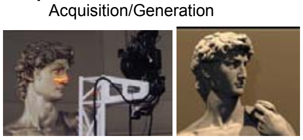
An Example



University of British Columbia

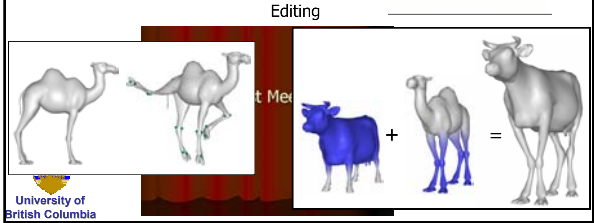
Operations

Acquisition/Generation



Transmission/Storage

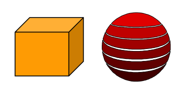
Editing

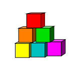


University of British Columbia

Geometry

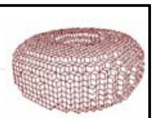
- Mathematical models of real world objects shape
- Categories:
  - Boundary representations
    - Freeform
    - Meshes & subdivision
  - Volumetric representations
    - Primitive based
    - Voxels



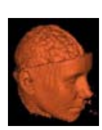


University of British Columbia

Volumetric - Volxelization



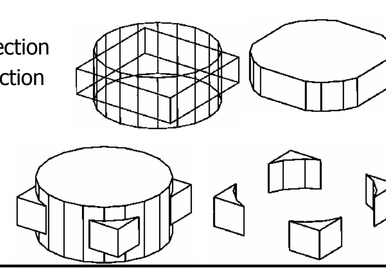
- Voxel based
  - Space subdivided into equal size boxes – each has in/out flag
  - Common in imaging applications (CT,MRI,Ultrasound)
- Extension – Octree (recursive construction)
- To draw use *iso-surfaces* – boundary between voxels with different flag



University of British Columbia

Volumetric - Primitives

- Use set of volumetric primitives
  - Box, sphere, cylinder, cone, etc…
- For complex objects use boolean operations
  - Union
  - Intersection
  - Subtraction



University of British Columbia

Freeform Representation

- Explicit form: z=z(x,y)
- Implicit form: f(x,y,z)=0
- Parametric form: [x(u,v),y(u,v),z(u,v)]
- Example – origin centered sphere of radius *R:*

Explicit is a special case of implicit and parametric form

Explicit:

$$z = +\sqrt{R^2 - x^2 - y^2} \cup z = -\sqrt{R^2 - x^2 - y^2}$$

Implicit:

$$x^2 + y^2 + z^2 - R^2 = 0$$

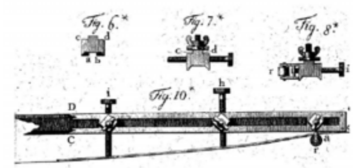
Parametric:

$$(x,y,z) = (R\cos\theta\cos\psi, R\sin\theta\cos\psi, R\sin\psi), \theta \in [0,2\pi], \psi \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

University of British Columbia

Splines – Free Form Curves

- Description = basis functions + coefficients
- Geometric meaning of coefficients (base)
  - Approximate/interpolate set of positions, derivatives, etc..



- Usually parametric

University of British Columbia

Polynomial Bases

- Monomial basis
  - Geometrically meaningless $\{1, t, t^2, t^3, \ldots\}$
- Other bases
  - Bezier
  - Hermitte
  - Lagrange, B-Spline, …
- Number of coefficients = polynomial rank
- Geometric meaning
  - coefficients - positions/derivatives, etc…
- Advantage
  - easy to analyze, derivatives remain polynomial

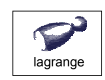
University of British Columbia

Lagrange

- Interpolates ***control points***
  - $P_0, \ldots, P_n$
- Base function $P_i(t)$ per control-point P_i
  - $P_i(t) = 1$ at i/n
  - $P_i(t) = 0$ at j/n, $j \,!= i$
- Each base function is a polynomial of degree n

$$P_i(t) = a_0^i + a_1^i t + a_2^i t^2 + \cdots + a_n^i t^n$$

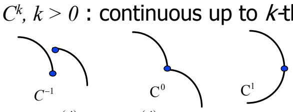
- Uniqueness
  - n equations in n unknowns (a^i_j)



University of British Columbia

Analytic Continuity

- $C_1(t)$ & $C_2(t)$, $t \in [0,1]$ - parametric curves
- Level of continuity at $C_1(1)$ and $C_2(0)$ is:
  - C^{-1}: $C_1(1) \neq C_2(0)$ (discontinuous)
  - C^0: $C_1(1) = C_2(0)$ (positional continuity)
  - C^k, $k > 0$: continuous up to *k*-th derivative



$$C_1^{(j)}(1) = C_2^{(j)}(0), 0 \le j \le k$$

- Continuity of single curve defined similarly
  - for polynomial bases C^∞

University of British Columbia

Geometric Continuity

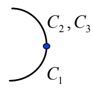
- Analytic continuity - too strong a requirement
- Geometric continuity – common curve is geometrically smooth (per given level *k*)
  - G^k, $k \le 0$: Same as C^k
  - G^k $k = 1$: $C'_1(1) = \alpha C'_2(0)$ $\alpha > 0$
  - G^k $k \ge 0$: In arc-length reparameterization of $C_1(t)$ & $C_2(t)$, the two are C^k

University of British Columbia

Geometric Continuity

- E.g.

$$C_1(t) = [\cos(t), \sin(t)] \quad t \in [-0.5\pi, 0]$$
$$C_2(t) = [\cos(t), \sin(t)] \quad t \in [0, 0.5\pi]$$
$$C_3(t) = [\cos(2t), \sin(2t)] \quad t \in [0, 0.25\pi]$$



  - $C_1(t)$ & $C_2(t)$ are C^k (& G^k) continuous
  - $C_1(t)$ & $C_3(t)$, are G^k continuous (not C^k)

University of British Columbia

Hermite Cubic Basis

- Geometrically-oriented basis for cubic polynomials
  - 2 positions + 2 tangents
- Has to satisfy

$$h_{i,j}(t): i,j = 0,1, \; t \in [0,1]$$

curve	$h(0)$	$h(1)$	$h'(0)$	$h'(1)$
$h_{00}(t)$	1	0	0	0
$h_{01}(t)$	0	1	0	0
$h_{10}(t)$	0	0	1	0
$h_{11}(t)$	0	0	0	1

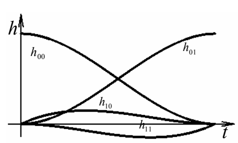
University of British Columbia

Hermite Cubic Basis (cont'd)

- Four cubics which satisfy the conditions

$$h_{00}(t) = t^2(2t-3)+1 \qquad h_{01}(t) = -t^2(2t-3)$$
$$h_{10}(t) = t(t-1)^2 \qquad h_{11}(t) = t^2(t-1)$$

- Obtain - solve 4 linear equations in 4 unknowns for each basis function



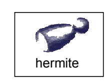
University of British Columbia

Hermite Cubic Basis (cont'd)

- Let C(t) be a cubic polynomial defined as the linear combination:

$$C(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t)$$

- What are C(0), C(1), C'(0), C'(1) ?



- $C(0)=P_0$, $C(1) = P_1$, $C'(0)=T_0$, $C'(1)=T_1$
- To generate a curve through P_0 & P_1 with slopes T_0 & T_1 use

$$C(t) = P_0 h_{00}(t) + P_1 h_{01}(t) + T_0 h_{10}(t) + T_1 h_{11}(t)$$

University of British Columbia

Natural Cubic Splines

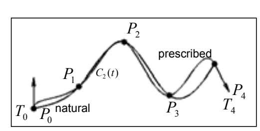
- Standard spline input – set of points $\{P_i\}_{i=0}^n$
  - No derivatives
- Interpolate by n cubic segments:
  - Derive $\{T_i\}_{i=0}^n$ from continuity constraints
  - Solve 4n equations

Interpolation ($2n$ equations):
$C_i(0) = P_{i-1} \quad C_i(1) = P_i \quad i = 1,..,n$
C^1 continuity constraints ($n-1$ equations):
$C_i'(1) = C_{i+1}'(0) \qquad i = 1,..,n-1$
C^2 continuity constraints ($n-1$ equations):
$C_i''(1) = C_{i+1}''(0) \qquad i = 1,..,n-1$

University of British Columbia

Natural Cubic Splines

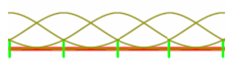
- Need another 2 equations to reach 4n
- Options
  - Natural end conditions: $C_1''(0) = 0, C_n''(1) = 0$
  - Prescribed end conditions (derivative available): $C_1'(0) = T_0, C_n'(1) = T_n$



University of British Columbia

B-Splines

- Idea: Generate basis where functions are continuous cross domains



- Control point controls set of basis functions (to preserve continuity)
- Alternative view: continuous basis functions defined on several domains

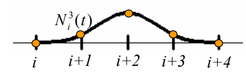
University of British Columbia

Uniform Cubic B-Spline Curves

- Definition

$$C(t)=\sum_{i=0}^{n-1} P_i N_i^3(t)\quad t\in[3,n]$$

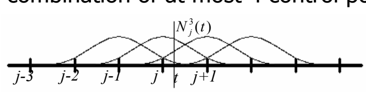
$$N_i^3(t)=\begin{cases} r^3/6 & r=t-i \quad t\in[i,i+1] \\ (-3r^3+3r^2+3r+1)/6 & r=t-i-1 \quad t\in[i+1,i+2] \\ (3r^3-6r^2+4)/6 & r=t-i-2 \quad t\in[i+2,i+3] \\ (1-r)^3/6 & r=t-i-3 \quad t\in[i+3,i+4] \end{cases}$$



University of British Columbia

Uniform Cubic B-Spline Curves

- For any t ∈ [3,n] $\sum_{i=j-3}^{j} N_i^3(t)=1$
- For any t ∈ [j,j+1] only 4 basis functions are non zero
- Any point on cubic B-Spline is affine combination of at most 4 control points



University of British Columbia

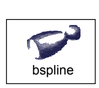
Boundary Conditions for B-Splines

- B-Splines do not interpolate any control points
  - in particular end points
- Way to force endpoint interpolation:
  - Let $P_0=P_1=P_2$ and same for other end
- Question:
  - What is the shape of the curve at endpoints if this method is used ?

University of British Columbia

B-Spline Curve Properties

- For n control points, C(t) is a piecewise polynomial of degree 3, defined for
- $C(t)\in\bigcup_{i=0}^{n-3} CH(P_i,\ldots,P_{i+3})\quad t\in[3,n]$
- C(t) is affine invariant
- Questions:
  - What is C(i) ?
  - What is C′(i) ?
  - What is the continuity of C(t) ?



University of British Columbia

NURBs

- B-Spline

$$C(t)=\sum_{i=0}^{n-1} P_i N_i^3(t)\quad t\in[3,n]$$

$$N_i^3(t)=\begin{cases} r^3/6 & r=t-i \quad t\in[i,i+1] \\ (-3r^3+3r^2+3r+1)/6 & r=t-i-1 \quad t\in[i+1,i+2] \\ (3r^3-6r^2+4)/6 & r=t-i-2 \quad t\in[i+2,i+3] \\ (1-r)^3/6 & r=t-i-3 \quad t\in[i+3,i+4] \end{cases}$$

- Non-Uniform – different interval lengths (knots)
- Rational – rational basis functions

$$C(t)=\frac{\sum_{i=0}^{n-1} w_i P_i N_i^3(t)}{\sum_{i=0}^{n-1} w_i N_i^3(t)}\quad t\in[3,n]$$

University of British Columbia

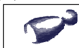
From Curves to Surfaces

- Curve is expressed as inner product of P_i coefficients and basis functions

$$C(u) = \sum_{i=0}^{n} P_i B_i(u)$$

- To extend curves to surfaces - treat surface as a curve of curves
- Assume P_i is not constant, but a function of second parameter v: $P_i(v) = \sum_{j=0}^{m} Q_{ij} B_j(v)$

$$C(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} Q_{ij} B_j(v) B_i(u)$$



bezpatch

University of British Columbia

Bilinear Patches

- Bilinear interpolation of 4 3D points - 2D analog of 1D linear interpolation between 2 points in the plane
- Given $P_{00}, P_{01}, P_{10}, P_{11}$ associated parametric bilinear surface for $u, v \in [0,1]$ is:

$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$

- Questions:
  - What does an isoparametric curve of a bilinear patch look like ?
  - When is a bilinear patch planar?

University of British Columbia