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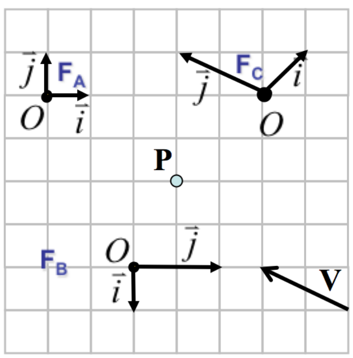
Worth 7% (48 points)

1. Transformations as a change of coordinate frame



$$\begin{bmatrix} \\ \\ 1 \end{bmatrix}_A = \begin{array}{ccc} i_B & j_B & O_B \\ \end{array} \begin{bmatrix} 0 & 2 & 2 \\ -1 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \\ \\ 1 \end{bmatrix}_B$$

$\langle 0,-1 \rangle$ $i_B = -j_A$
$\langle 2,0 \rangle$ $j_B = 2i_A$
$(2,-4)$ $O_B = O_A + 2i_A - 4j_A$

$$\begin{bmatrix} \\ \\ 1 \end{bmatrix}_C = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{5}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \\ \\ 1 \end{bmatrix}_A$$

$i_A = (i_C - j_C)/3$
$j_A = (2i_C + j_C)/3$
$O_A = O_C - \frac{5}{3}(i_C - j_C)$

(a) (3 points) Express the coordinates of point P with respect to coordinate frames A, B, and C.

$P = O_A + 3i_A - 2j_A \Rightarrow P_A(3,-2)$
$P = O_B - 2i_B + 0.5j_B \Rightarrow P_B(-2,0.5)$
$P = O_C - 2i_C \Rightarrow P_C(-2,0)$

(b) (3 points) Express the coordinates of vector V with respect to coordinate frames A, B, and C.

$V = -2i_A + j_A \Rightarrow V_A\langle -2,1 \rangle$
$V = -i_B - j_B \Rightarrow V_B\langle -1,-1 \rangle$
$V = j_C \Rightarrow V_C\langle 0,1 \rangle$

(c) (3 points) Fill in the 2D transformation matrix that takes points from F_B to F_A, as given to the right of the above figure.

(d) (3 points) Fill in the 2D transformation matrix that takes points from F_A to F_C, as given to the right of the above figure. Hint: You might first want to determine what combination of i_C and j_C are needed to reproduce $3i_A$ and $3j_A$.

(e) (2 points) Using your answers for the above two matrices, develop a 2D transformation matrix that takes points from F_B to F_C. Test your solution using point P.

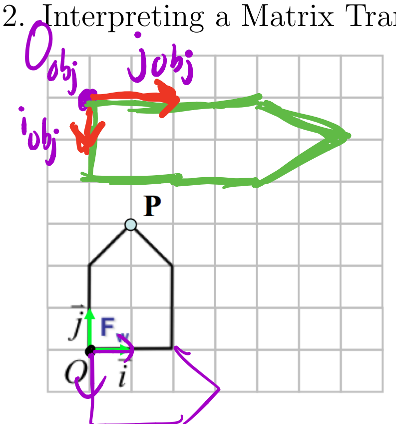
$P_C = M_{A\to C} \cdot M_{B\to A} \cdot P_B$

$$= \frac{1}{3}\begin{bmatrix} 1 & 2 & -5 \\ -1 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 \\ -1 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{4}{3} + \frac{1}{3} - \frac{11}{3} = -2$

$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}_C = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{11}{3} \\ -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ \frac{1}{2} \\ 1 \end{bmatrix}_B$$

2. Interpreting a Matrix Transformation



$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_w = \begin{array}{c} \begin{array}{ccc} i_{obj} & j_{obj} & O_{obj} \end{array} \\ \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 6 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{obj}$$

(a) (2 points) On the above diagram, sketch the origin and basis vectors of the coordinate frame F_{obj} that results from the given transformation matrix.

(b) (2 points) The drawing of the above house represents its untransformed shape. Sketch the transformed version of the house in the above diagram.

(c) (2 points) Given $M = Translate(a, b, 0)Rotate(z, \theta)Scale(c, d, 1)$, provide the values of a, b, c, d and θ that would implement the given transformation.

$M = Translate(0,6,0)\ Rotate(z,-90^\circ)\ Scale(1,2,0)$

$a = 0$ $\quad c = 1$

$b = 6$ $\quad d = 2$

$\theta = -90^\circ$

(d) (2 points) Given $M = Rotate(z, \theta)Translate(a, b, 0)Scale(c, d, 1)$, provide the values of a, b, c, d and θ that would implement the given transformation.

$M = Rotate(z,-90^\circ)\ Translate(-6,0,0)\ Scale(1,2,0)$

$a = -6$

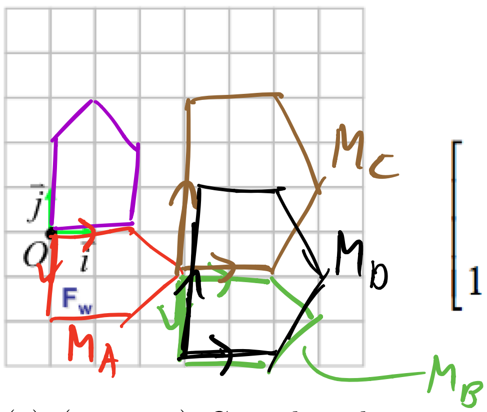
$b = 1$

$\theta = -90^\circ$

$c = 1$

$d = 2$

3. Composing Transformations



$$\begin{bmatrix} \\ \\ 1 \end{bmatrix}_w = \begin{array}{c} \begin{array}{ccc} \vec{i}_{obj} & \vec{j}_{obj} & O_{obj} \end{array} \\ \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} \\ \\ 1 \end{bmatrix}_{obj}$$

(a) (3 points) Consider a house in the xy-plane, defined by the coordinates (0,0), (2,0), (2,2), (1,3), 0,2). Assume $z = 0$ for all vertices.
Sketch the untransformed house, as well as the transformed house that would result after each step of the following sequence of transformations. Assume that the matrix `M` is initialised to the identity matrix, and that all the transformations right multiply the current value of `M` by the given transformation.

```
M.rotate(z,-90);        // A
M.translate(1,3,0);     // B
M.scale(-2,1,1);        // C
M.translate(-1,0,0);    // D
```

(b) (3 points) Give the final resulting 3×3 transformation matrix, `M`; use the space to the right of the diagram above. The matrix will be easiest to develop by directly observing the final coordinate frame after step D of part (a).

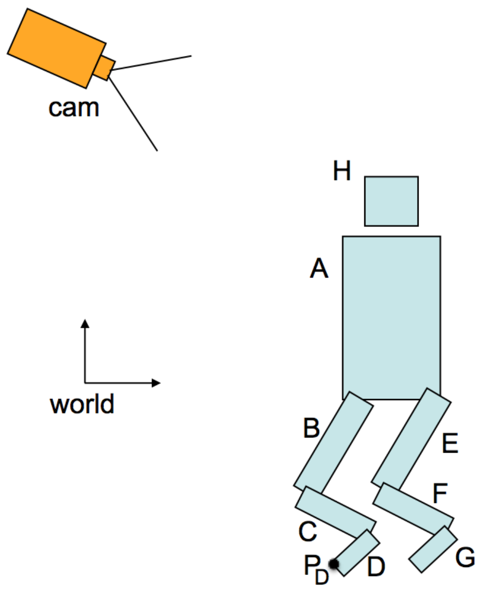
(c) (2 points) Assume that the matrices for the individual transformation steps are labelled M_A, M_B, M_C, M_D. Give an algebraic expression for the final resulting transformation matrix, M. Also, give an algebraic expression for M^{-1}.

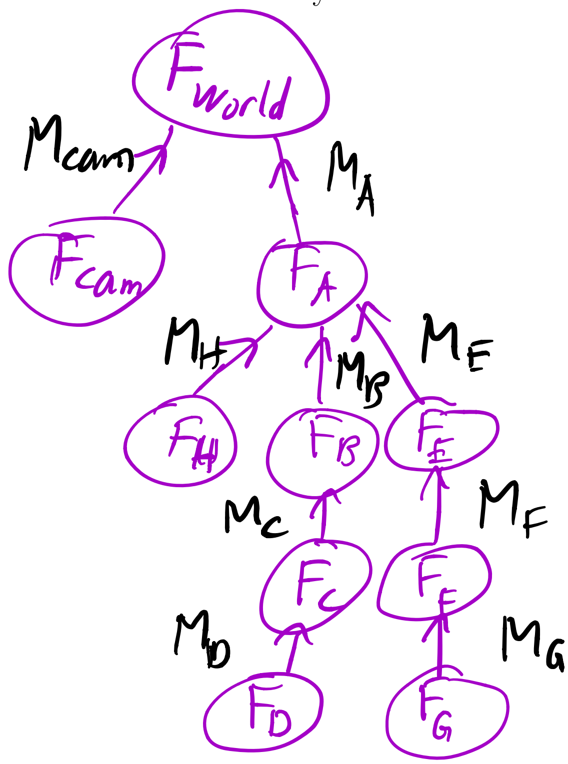
$$M = M_A M_B M_C M_D$$

$$M^{-1} = M_D^{-1} M_C^{-1} M_B^{-1} M_A^{-1}$$

4. Scene Graphs

(a) (3 points) Sketch a scene graph for the armless-robot scene below. The labeled nodes should represent the coordinate frames, e.g., F_A, for each link (not explicitly shown in the diagram). Use directed edges, i.e., arrows, to represent transformations that connect the nodes in the graph. Label each edge with a unique name, e.g., M_A, which indicates the transformation matrix that takes points from the given frame to its parent frame. Use the world coordinate frame as the root note of the scene graph. Assume that the body, F_A, and the camera, F_{cam}, are positioned relative to the world frame, F_{world}. Assume that all other parts are defined relative to the links that their proximal parent links, i.e., the links closer to the body.





(b) (2 points) Give an algebraic expression for the composite transformation that would be used when drawing a point P_D, as defined in frame F_D. i.e., it should transform point P_D to the camera coordinate frame, F_{cam}. Your answer should be expressed as a product of the matrices used to label your scene graph.

$$P_{cam} = M_{cam}^{-1} M_A M_B M_C M_D P_D$$

(c) (2 points) Similarly, give a algebraic expression for the composite transformation that transforms point, P_D, as defined in frame F_D, to the head frame, F_H.

$$P_H = M_H^{-1} \cdot M_B \cdot M_C \cdot M_D \cdot P_D$$

5. A requirement of a 3×3 rotation matrix, R, is that the columns have unit magnitude and that are mutually orthogonal, i.e., a zero dot product.

   (a) (1 point) Given a rotation matrix defined by three column vectors, $R = \begin{bmatrix} \vec{\mathbf{d}} & \vec{\mathbf{e}} & \vec{\mathbf{f}} \end{bmatrix}$ compute the resulting matrix product, $M = R^T R$.

$$\begin{bmatrix} d \\ e \\ f \end{bmatrix} \cdot \begin{bmatrix} d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{3x3}$$

   (b) (2 points) A 4×4 rigid body transformation is defined by a rotation matrix and a translation, T, as shown below. Develop an expression for the inverse of this transformation matrix. Hint: your answer to part (a) provides most of the solution.

$$M = \begin{bmatrix} d_x & e_x & f_x & T_x \\ d_y & e_y & f_y & T_y \\ d_z & e_z & f_z & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M^{-1} \rightarrow \begin{bmatrix} d & -T \cdot d \\ e & -T \cdot e \\ f & -T \cdot f \\ 0 \; 0 \; 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} d & e & f & T \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_{4x4}$$

   (c) (4 points) Determine if the matrices below are rotations. Why or why not? If it is a rotation, describe it, i.e., `Rotate(`z`,`θ`)`. If not, is there another requirement that a rotation matrix should satisfy, in addition to being orthonormal (unit-length, mutually orthogonal columns)?

   (i) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   no yes no

$|i| = 1 \quad i \cdot j = 0$

$|j| = 1 \quad i \cdot k = 0$

$|k| = 1 \quad j \cdot k = 0$

and: $i \times j = k$

There also mirror the object.

6. (4 points) Viewing Transformation

Determine the viewing transformation, M_{view}, that takes points from WCS (world coordinates) to VCS (viewing or camera coordinates), for the following camera parameters: $P_{eye} = (10, -50, 10)$, $P_{ref} = (10, 0, 10)$, $V_{up} = (0, 0, 1)$.

$$\vec{K} = P_{eye} - P_{ref} = \langle 0, -50, 0 \rangle$$

$$\vec{k} = \vec{K} / \|\vec{K}\| = \langle 0, -1, 0 \rangle$$

$$\vec{i} = \vec{V}_{up} \times \vec{k} = \langle 1, 0, 0 \rangle$$

$$\vec{j} = \vec{k} \times \vec{i} = \langle 0, 0, 1 \rangle$$

$$M_{cam} = \begin{array}{cccc} & & & \\ \left[\begin{matrix} 1 & 0 & 0 & 10 \\ 0 & 0 & -1 & -50 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{matrix}\right] \\ i_{cam} \quad j_{cam} \quad k_{cam} \quad O_{cam} \end{array}$$

$$M_{view} = M_{cam}^{-1}$$

(optional)

$$= \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 0 & 1 & -10 \\ 0 & -1 & 0 & 50 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$