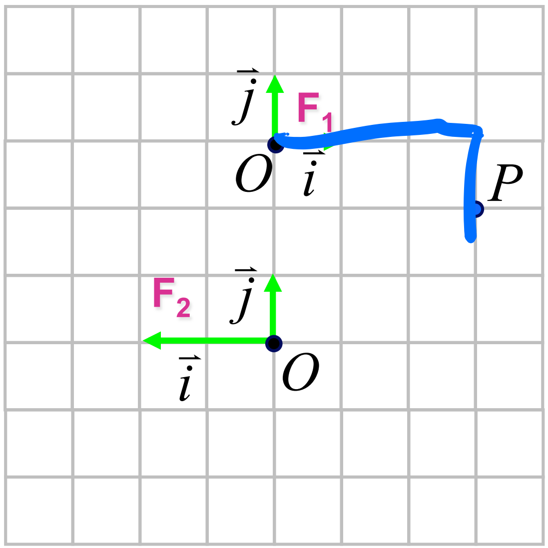
Transformations as a change of basis



$P_1 = (3,-1)$ $P_2 = (-1.5, 2)$ **Goal:** $P_2 = M P_1$

$$P = O + x\vec{i} + y\vec{j}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_1 + x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}_1 + y_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}_1$$

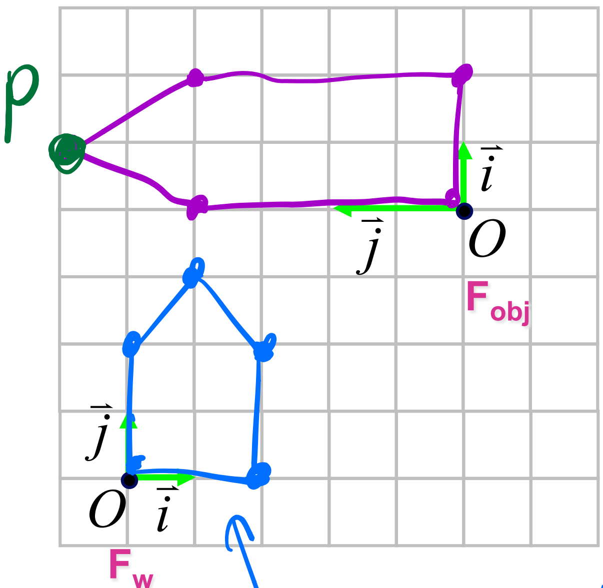
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + x_1 \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}_2 + y_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}_2$$

$$\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

check:

$$\begin{bmatrix} -1.5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Transformations as a change of basis



Goal: $P_w = M P_{obj}$

$$\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} = \begin{array}{ccc} i_{obj} & j_{obj} & O_{obj} \\ \end{array} \begin{bmatrix} 0 & -2 & 5 \\ 1 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}}_{P_{obj}}$$

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

untransformed house, i.e., $F_w = F_h$

$$P_{obj} = O_{obj} + x_{obj}\, i_{obj} + y_{obj}\, j_{obj}$$

3D Transformations

Affine transformations

- linear transformation + translations
- can be expressed as a 3x3 matrix + 3 vector

$$P' = M \cdot P + T$$

4x4 matrices

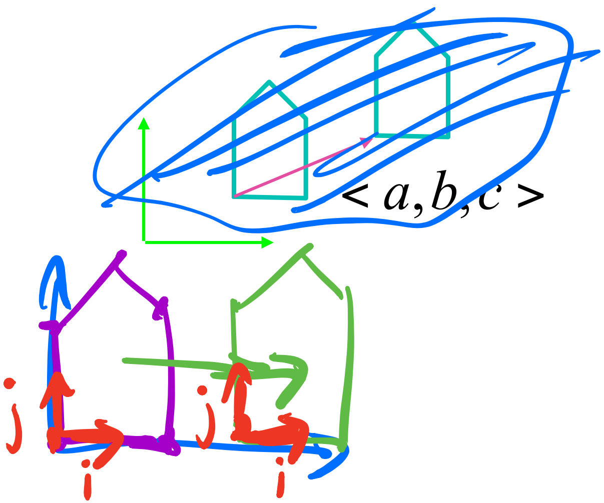
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{array}{cccc} i & j & k & O \\ \multicolumn{4}{c}{\left[\begin{matrix} m_{11} & m_{12} & m_{13} & T_x \\ m_{21} & m_{22} & m_{23} & T_y \\ m_{31} & m_{32} & m_{33} & T_z \\ 0 & 0 & 0 & 1 \end{matrix}\right]} \end{array} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

h=1

$$P = O + x\vec{i} + y\vec{j} + z\vec{k}$$

Transformations

Translation



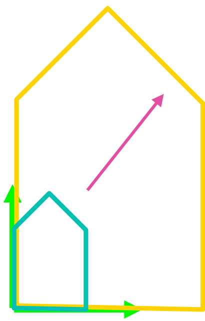
Translate(a,b,c)
Trans(a,b,c)

$$x' = x + a$$
$$y' = y + b$$
$$z' = z + c$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & & a \\ & 1 & & b \\ & & 1 & c \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Transformations

Scaling



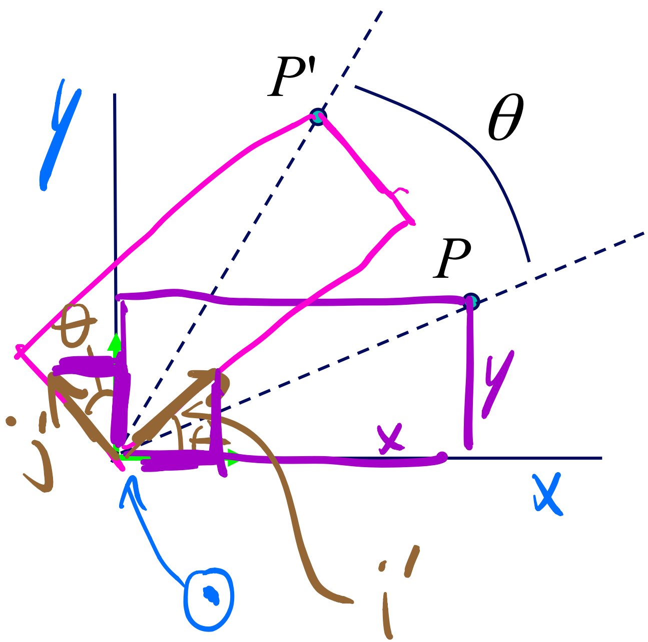
Scale(a,b,c)

$$x' = a x$$
$$y' = b y$$
$$z' = c z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Transformations

Rotation



$Rotate(z,\theta)$

$Rot(z,\theta)$

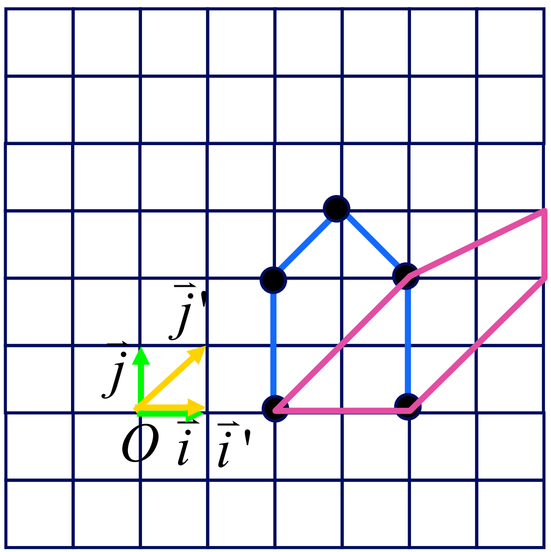
CCW is +ve

$$\begin{bmatrix} i' & j' & k' & o' \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

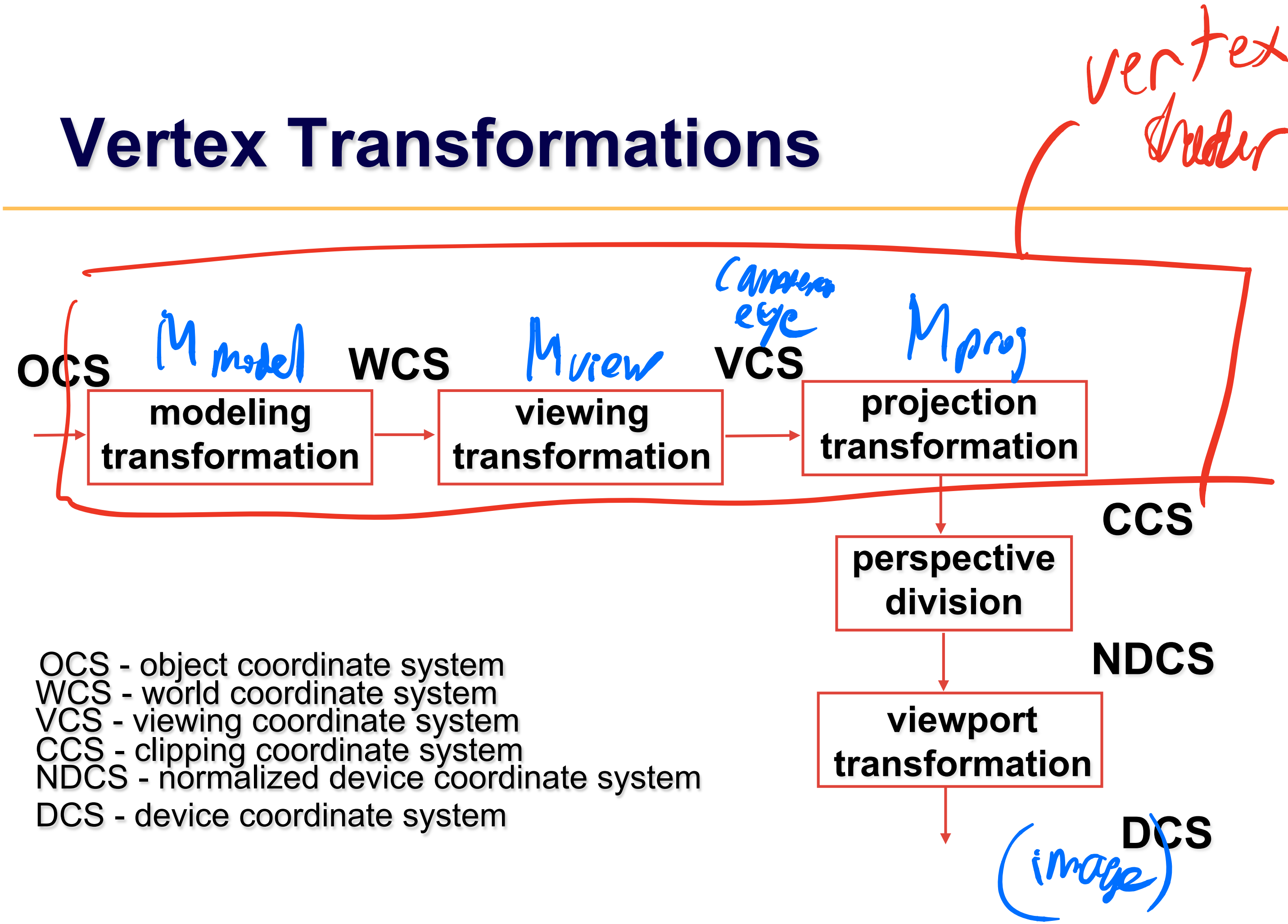
Transformations

Shear



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Vertex Transformations



OCS - object coordinate system
WCS - world coordinate system
VCS - viewing coordinate system
CCS - clipping coordinate system
NDCS - normalized device coordinate system
DCS - device coordinate system

Composition of Transformations

reminder:

translate(a,b,c)

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & & a \\ & 1 & & b \\ & & 1 & c \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

scale(a,b,c)

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$Rotate(z,\theta)$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

or build 4x4 matrix directly

$$\begin{bmatrix} i & j & k & o \\ 0 & 0 & 0 & 1 \end{bmatrix} \Big\} \ i, j, k, O \text{ of obj expressed wrt world.}$$

Simple Compositions

translate(a,b,c) translate(d,e,f) $= \text{translate}(a+d, b+e, c+f)$

$$\begin{bmatrix} 1 & & & a \\ & 1 & & b \\ & & 1 & c \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & d \\ & 1 & & e \\ & & 1 & f \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & a+d \\ & 1 & & b+e \\ & & 1 & c+f \\ & & & 1 \end{bmatrix}$$

scale(a,b,c) scale(d,e,f)

$$\begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & 1 \end{bmatrix}\begin{bmatrix} d & & & \\ & e & & \\ & & f & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} ad & & & \\ & be & & \\ & & cf & \\ & & & 1 \end{bmatrix} = \text{scale}(ad, be, cf)$$

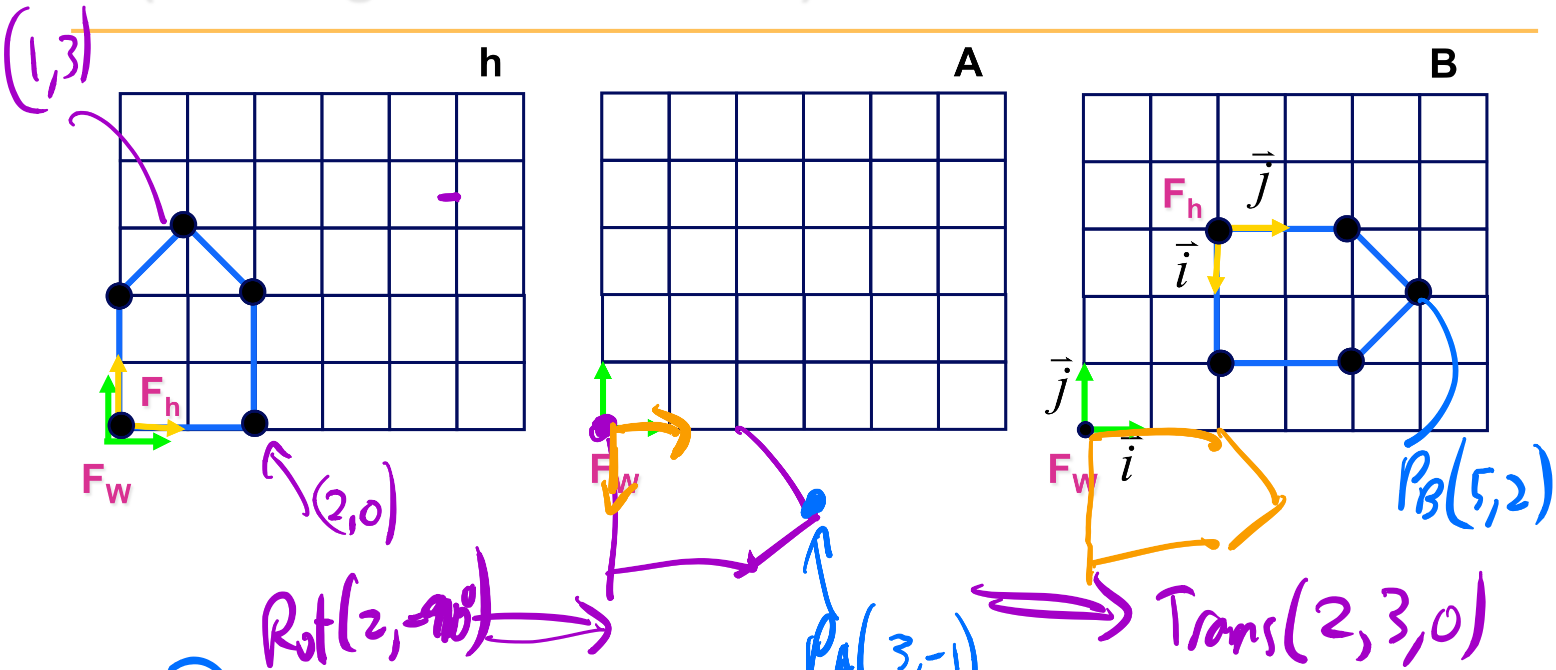
$Rotate(z,\theta_1)\ Rotate(z,\theta_2)$

$$\begin{bmatrix} C_1 & -S_1 & & \\ S_1 & C_1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} C_2 & -S_2 & & \\ S_2 & C_2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} C_1C_2 - S_1S_2 & -S_1C_2 - C_1S_2 & & \\ S_1C_2 + C_1S_2 & C_1C_2 - S_1S_2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$C_{12} = \cos(\theta_1 + \theta_2)$

$S_{12} = \sin(\theta_1 + \theta_2)$

Composing Transformations
(thinking in fixed coords)

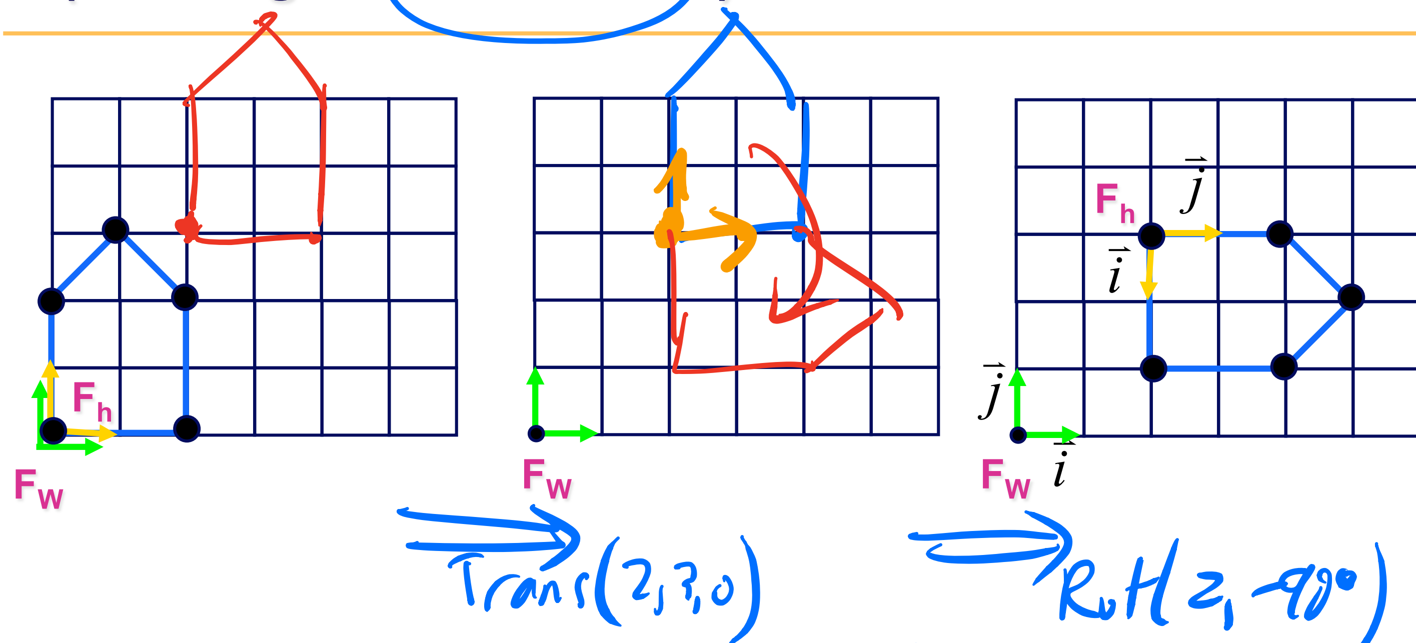


① $P_A = Rot(z,-90)P_h$

② $P_B = Trans(2,3,0)P_A$

$P_B = Trans(2,3,0)\,Rot(z,-90°)\,P_h$

Composing Transformations
(thinking in local coords)



$P_W = \text{Trans}(2,3,0)\,\text{Rot}(z,-90°)\,P_h$

right multiply

Composing Transformations

(a) $P_w = Trans(2,3,0)\, Rot(z,-90°)\, P_h$ prev two slides

or (b) $P_w = Rot(z,-90°)\, Trans(-3,2,0)\, P_h$

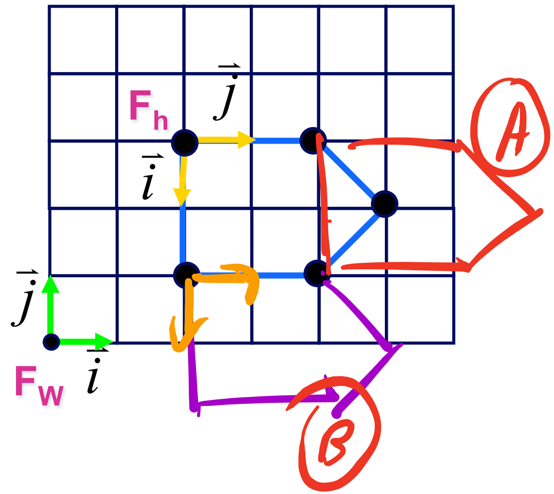
- left multiply: R-to-L
  - *interpret operations wrt fixed coords*
- right multiply: L-to-R (default for **code**)
  - *interpret operations wrt local coords*

(a)

```
M = I                      // M
M.translate(2,3,0);        // M ← M·T
M.rotate(z,-90°)           // M ← M·R
```

$\rightsquigarrow M = T \cdot R$

Summary Example



$T = Trans(2,0,0)$

$$P' = M\,P$$

left multiply A

right multiply B

$P' = T \cdot M \cdot P$

$P' = M \cdot T \cdot P$

$P' = MTP$