Slide 1

University of British Columbia
CPSC 314 Computer Graphics
Jan-Apr 2013

Tamara Munzner

Math Review
guest lecture: James Gregson

Week 1, Wed Jan 2

http://www.ugrad.cs.ubc.ca/~cs314/Vjan2013

Slide 2 — News

- usual lecture order switched
  - math review today (Jan 2)
    - James Gregson, TA
    - usually second lecture
  - course intro and overview Friday (Jan 4)
    - Tamara Munzner, instructor
    - usually first lecture

Slide 3 — Notation: Scalars, Vectors, Matrices

- scalar $\quad a$
  - (lower case, italic)
- vector $\quad \mathbf{a} = \begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix}$
  - (lower case, bold)
- matrix
  - (upper case, bold)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Slide 4 — Vectors

- arrow: length and direction
  - oriented segment in nD space
- offset / displacement
  - location if given origin
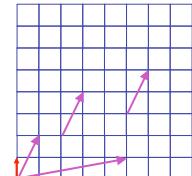
Slide 5 — Column vs. Row Vectors

- row vectors $\quad \mathbf{a}_{row} = \begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix}$

- column vectors $\quad \mathbf{a}_{col} = \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix}$

- switch back and forth with transpose

$$\mathbf{a}_{col}^T = \mathbf{a}_{row}$$

Slide 6 — Vector-Vector Addition

- add: vector + vector = vector
- parallelogram rule
  - tail to head, complete the triangle

geometric   algebraic

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

examples: $\quad (3,2) + (6,4) = (9,6)$
$\quad (2,5,1) + (3,1,-1) = (5,6,0)$

Slide 7 — Vector-Vector Subtraction

- subtract: vector - vector = vector

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}$$

$\mathbf{u} - \mathbf{v} =$
$\mathbf{u} + (-\mathbf{v})$

$(3,2) - (6,4) = (-3,-2)$
$(2,5,1) - (3,1,-1) = (-1,4,2)$

Slide 8 — Vector-Vector Subtraction

- subtract: vector - vector = vector

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}$$

$\mathbf{u} - \mathbf{v} =$
$\mathbf{u} + (-\mathbf{v})$

$(3,2) - (6,4) = (-3,-2)$
$(2,5,1) - (3,1,-1) = (-1,4,2)$

argument reversal

Slide 9 — Scalar-Vector Multiplication

- multiply: scalar * vector = vector
  - vector is scaled

$a * \mathbf{u} = (a*u_1, a*u_2, a*u_3)$

$2*(3,2) = (6,4)$
$.5*(2,5,1) = (1,2.5,.5)$

Slide 10 — Vector-Vector Multiplication

- multiply: vector * vector = scalar
- dot product, aka inner product $\quad \mathbf{u} \bullet \mathbf{v}$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1*v_1) + (u_2*v_2) + (u_3*v_3)$$

Slide 11 — Vector-Vector Multiplication

- multiply: vector * vector = scalar
- dot product, aka inner product $\quad \mathbf{u} \bullet \mathbf{v}$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1*v_1) + (u_2*v_2) + (u_3*v_3)$$

Slide 12 — Vector-Vector Multiplication

- multiply: vector * vector = scalar
- dot product, aka inner product $\quad \mathbf{u} \bullet \mathbf{v}$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1*v_1) + (u_2*v_2) + (u_3*v_3)$$

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

- geometric interpretation
  - lengths, angles
 - can find angle between two vectors

Slide 13 — Dot Product Geometry

- can find length of projection of u onto v

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

$$\|\mathbf{u}\|\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|}$$

- as lines become perpendicular, $\mathbf{u} \bullet \mathbf{v} \to 0$

Slide 14 — Dot Product Example

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1*v_1) + (u_2*v_2) + (u_3*v_3)$$

$$\begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} = (6*1) + (1*7) + (2*3) = 6 + 7 + 6 = 19$$

Slide 15 — Vector-Vector Multiplication, The Sequel

- multiply: vector * vector = vector
- cross product
  - algebraic

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Slide 16 — Vector-Vector Multiplication, The Sequel

- multiply: vector * vector = vector
- cross product
  - algebraic

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

## Vector-Vector Multiplication, The Sequel

- multiply: vector * vector = vector
- cross product
  - algebraic

$$3 \quad 1 \quad 2 \quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

blah blah

17

Vector-Vector Multiplication, The Sequel

- multiply: vector * vector = vector
- cross product
  - algebraic

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

  - geometric

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin\theta$$

  - $\|\mathbf{a} \times \mathbf{b}\|$ parallelogram area
  - $\mathbf{a} \times \mathbf{b}$ perpendicular to parallelogram
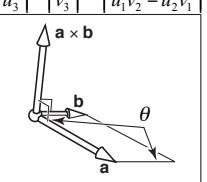
18

RHS vs. LHS Coordinate Systems

- right-handed coordinate system   convention
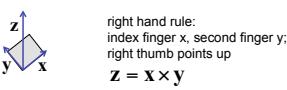
right hand rule:
index finger x, second finger y;
right thumb points up

$$\mathbf{z} = \mathbf{x} \times \mathbf{y}$$

- left-handed coordinate system
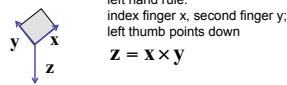
left hand rule:
index finger x, second finger y;
left thumb points down

$$\mathbf{z} = \mathbf{x} \times \mathbf{y}$$

19

Basis Vectors

- take any two vectors that are linearly independent (nonzero and nonparallel)
  - can use linear combination of these to define any other vector:

$$\mathbf{c} = w_1 \mathbf{a} + w_2 \mathbf{b}$$

$\mathbf{c} = 2\mathbf{a} + 0.5\mathbf{b}$

$0.5\mathbf{b}$

$2\mathbf{a}$
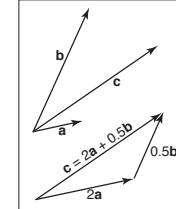
20

Orthonormal Basis Vectors

- if basis vectors are orthonormal (orthogonal (mutually perpendicular) and unit length)
  - we have Cartesian coordinate system
  - familiar Pythagorean definition of distance

orthonormal algebraic properties

$$\|\mathbf{x}\| = \|\mathbf{y}\| = 1,$$
$$\mathbf{x} \bullet \mathbf{y} = 0$$

$\mathbf{c} = 2\mathbf{x} + 0.5\mathbf{y}$

$0.5\mathbf{y}$

$2\mathbf{x}$
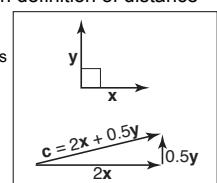
21

Basis Vectors and Origins

- coordinate system: just basis vectors
  - can only specify offset: vectors
- coordinate frame: basis vectors and origin
  - can specify location as well as offset: points
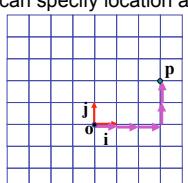
$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

22

Working with Frames
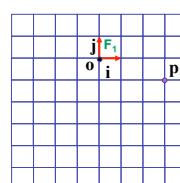
$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

F_1

23

Working with Frames
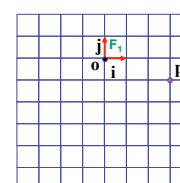
$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$F_1 \quad \mathbf{p} = (3,-1)$

24

Working with Frames
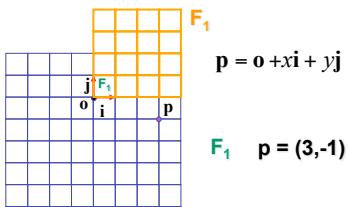
$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$F_1 \quad \mathbf{p} = (3,-1)$

25

Working with Frames

$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$F_1 \quad \mathbf{p} = (3,-1)$

F_2
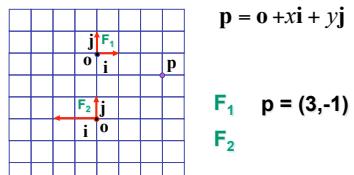
26

Working with Frames
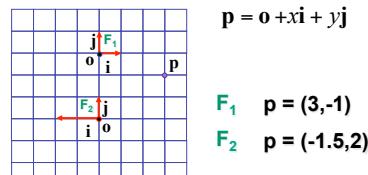
$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$F_1 \quad \mathbf{p} = (3,-1)$

$F_2 \quad \mathbf{p} = (-1.5,2)$

27

Working with Frames

F_2
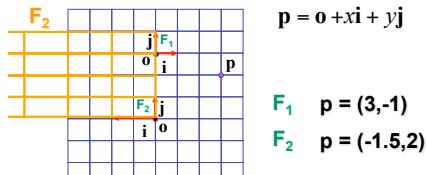
$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$F_1 \quad \mathbf{p} = (3,-1)$

$F_2 \quad \mathbf{p} = (-1.5,2)$

28

Working with Frames

$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$F_1 \quad \mathbf{p} = (3,-1)$

$F_2 \quad \mathbf{p} = (-1.5,2)$

F_3

29

Working with Frames

$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$F_1 \quad \mathbf{p} = (3,-1)$

$F_2 \quad \mathbf{p} = (-1.5,2)$

$F_3 \quad \mathbf{p} = (1,2)$

30

Working with Frames

F_3

$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$F_1 \quad \mathbf{p} = (3,-1)$

$F_2 \quad \mathbf{p} = (-1.5,2)$

$F_3 \quad \mathbf{p} = (1,2)$
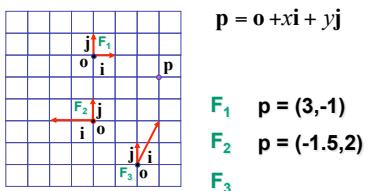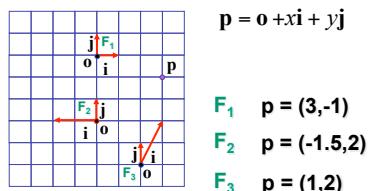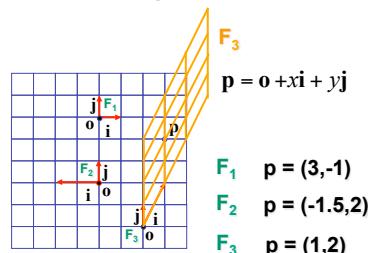
31

Named Coordinate Frames

- origin and basis vectors $\mathbf{p} = \mathbf{o} + a\mathbf{x} + b\mathbf{y} + c\mathbf{z}$
- pick canonical frame of reference
  - then don't have to store origin, basis vectors
  - just $\mathbf{p} = (a,b,c)$
  - convention: Cartesian orthonormal one on previous slide
- handy to specify others as needed
  - airplane nose, looking over your shoulder, ...
  - really common ones given names in CG
   - object, world, camera, screen, ...

32

Lines

- slope-intercept form
  - $y = mx + b$
- implicit form
  - $y - mx - b = 0$
  - $Ax + By + C = 0$
  - $f(x,y) = 0$

$f(x,y)=0$
$y=b$
$x=a$
$f(x,y) = y - mx - b$
$m = -b/a$
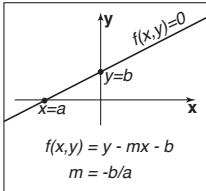
Implicit Functions

- find where function is 0
  - plug in (x,y), check if
    - 0: on line
    - < 0: inside
    - > 0: outside
- analogy: terrain
  - sea level: $f=0$
  - altitude: function value
  - topo map: equal-value contours (level sets)
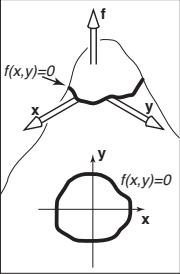
$f(x,y)=0$
$f(x,y)=0$

Implicit Circles

- $f(x, y) = (x - x_c)^2 + (y - y_c)^2 - r^2$
  - circle is points (x,y) where $f(x,y) = 0$
- $p = (x, y), c = (x_c, y_c) : (\mathbf{p} - \mathbf{c}) \bullet (\mathbf{p} - \mathbf{c}) - r^2 = 0$
  - points $\mathbf{p}$ on circle have property that vector from $\mathbf{c}$ to $\mathbf{p}$ dotted with itself has value r^2
- $\|\mathbf{p} - \mathbf{c}\|^2 - r^2 = 0$
  - points points $\mathbf{p}$ on the circle have property that squared distance from $\mathbf{c}$ to $\mathbf{p}$ is r^2
- $\|\mathbf{p} - \mathbf{c}\| - r = 0$
  - points $\mathbf{p}$ on circle are those a distance r from center point $\mathbf{c}$

Parametric Curves

- parameter: index that changes continuously
  - (x,y): point on curve
  - t: parameter
- vector form
  - $\mathbf{p} = f(t)$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g(t) \\ h(t) \end{bmatrix}$

2D Parametric Lines

- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 + t(x_1 - x_0) \\ y_0 + t(y_1 - y_0) \end{bmatrix}$
- $\mathbf{p}(t) = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0)$
- $\mathbf{p}(t) = \mathbf{o} + t(\mathbf{d})$

- start at point $\mathbf{p}_0$, go towards $\mathbf{p}_1$, according to parameter t
  - $\mathbf{p}(0) = \mathbf{p}_0$, $\mathbf{p}(1) = \mathbf{p}_1$

p(-1.0)
p(-0.5)
p(0.0) $\mathbf{p}_0$
p(0.25)
p(0.5)
p(1.0) $\mathbf{p}_1$
$\mathbf{p}_1$-$\mathbf{p}_0$
p(1.5)
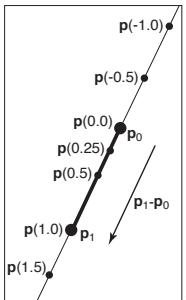
Linear Interpolation

- parametric line is example of general concept
  - $\mathbf{p}(t) = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0)$
- interpolation
  - $\mathbf{p}$ goes through $\mathbf{a}$ at $t = 0$
  - $\mathbf{p}$ goes through $\mathbf{b}$ at $t = 1$
- linear
  - weights t, $(1-t)$ are linear polynomials in t

Matrix-Matrix Addition

- add: matrix + matrix = matrix

$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} n_{11} + m_{11} & n_{12} + m_{12} \\ n_{21} + m_{21} & m_{22} + m_{22} \end{bmatrix}$

- example

$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 1+(-2) & 3+5 \\ 2+7 & 4+1 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 9 & 5 \end{bmatrix}$

Scalar-Matrix Multiplication

- multiply: scalar * matrix = matrix

$a \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} a*m_{11} & a*m_{12} \\ a*m_{21} & a*m_{22} \end{bmatrix}$

- example

$3 \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3*2 & 3*4 \\ 3*1 & 3*5 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 3 & 15 \end{bmatrix}$

Matrix-Matrix Multiplication

- can only multiply (n,k) by (k,m): number of left cols = number of right rows
  - legal

$\begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix} \begin{bmatrix} h & i \\ j & k \\ l & m \end{bmatrix}$

  - undefined

$\begin{bmatrix} a & b & c \\ e & f & g \\ o & p & q \end{bmatrix} \begin{bmatrix} h & i \\ j & k \end{bmatrix}$

Matrix-Matrix Multiplication

- row by column

$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$

$p_{11} = m_{11}n_{11} + m_{12}n_{21}$

Matrix-Matrix Multiplication

- row by column

$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$

$p_{11} = m_{11}n_{11} + m_{12}n_{21}$
$p_{21} = m_{21}n_{11} + m_{22}n_{21}$

Matrix-Matrix Multiplication

- row by column

$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$

$p_{11} = m_{11}n_{11} + m_{12}n_{21}$
$p_{21} = m_{21}n_{11} + m_{22}n_{21}$
$p_{12} = m_{11}n_{12} + m_{12}n_{22}$

Matrix-Matrix Multiplication

- row by column

$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$

$p_{11} = m_{11}n_{11} + m_{12}n_{21}$
$p_{21} = m_{21}n_{11} + m_{22}n_{21}$
$p_{12} = m_{11}n_{12} + m_{12}n_{22}$
$p_{22} = m_{21}n_{12} + m_{22}n_{22}$

Matrix-Matrix Multiplication

- row by column

$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$

$p_{11} = m_{11}n_{11} + m_{12}n_{21}$
$p_{21} = m_{21}n_{11} + m_{22}n_{21}$
$p_{12} = m_{11}n_{12} + m_{12}n_{22}$
$p_{22} = m_{21}n_{12} + m_{22}n_{22}$

- noncommutative: $\mathbf{AB} \mathrel{!=} \mathbf{BA}$

Matrix-Vector Multiplication

- points as column vectors: postmultiply

$\begin{bmatrix} x' \\ y' \\ z' \\ h' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ h \end{bmatrix}$ $\mathbf{p'} = \mathbf{Mp}$

- points as row vectors: premultiply

$\begin{bmatrix} x' & y' & z' & h' \end{bmatrix} = \begin{bmatrix} x & y & z & h \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}^T$ $\mathbf{p'}^T = \mathbf{p}^T \mathbf{M}^T$

Matrices

- transpose $\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}^T = \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix}$

- identity $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- inverse $\mathbf{AA}^{-1} = \mathbf{I}$
  - not all matrices are invertible

Matrices and Linear Systems

- linear system of n equations, n unknowns

$$3x + 7y + 2z = 4$$
$$2x - 4y - 3z = -1$$
$$5x + 2y + z = 1$$

- matrix form **Ax=b**

$$\begin{bmatrix} 3 & 7 & 2 \\ 2 & -4 & -3 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

Readings For Lecture

- FCG Chapter 2: Miscellaneous Math
  - except 2.7 (2.11 in 2nd edition)

- FCG Chapter 5: Linear Algebra
  - except 5.4 (not in 2nd edition)