University of British Columbia
CPSC 314 Computer Graphics
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Curves

http://www.ugrad.cs.ubc.ca/~cs314/Vjan2013

Reading

- FCG Chap 15 Curves
  - Ch 13 2nd edition

Curves

Parametric Curves

- parametric form for a line:

$$x = x_0 t + (1-t)x_1$$
$$y = y_0 t + (1-t)y_1$$
$$z = z_0 t + (1-t)z_1$$

- x, y and z are each given by an equation that involves:
  - parameter *t*
  - some user specified control points, x_0 and x_1
- this is an example of a parametric curve

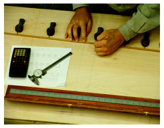
Splines

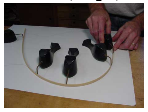
- a *spline* is a parametric curve defined by *control points*
  - term "spline" dates from engineering drawing, where a spline was a piece of flexible wood used to draw smooth curves
  - control points are *adjusted by the user* to control shape of curve

Splines - History

- draftsman used 'ducks' and strips of wood (splines) to draw curves
- wood splines have second-order continuity, pass through the control points

a duck (weight)





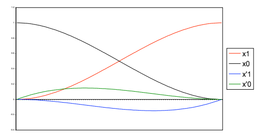
ducks trace out curve

Hermite Spline

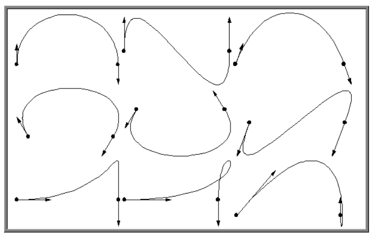
- *hermite spline* is curve for which user provides:
  - endpoints of curve
  - parametric derivatives of curve at endpoints
    - parametric derivatives are *dx/dt, dy/dt, dz/dt*
  - more derivatives would be required for higher order curves

Basis Functions

- a point on a Hermite curve is obtained by multiplying each control point by some function and summing
- functions are called *basis functions*

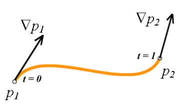


Sample Hermite Curves

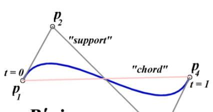


Bézier Curves

- similar to Hermite, but more intuitive definition of endpoint derivatives
- four control points, two of which are knots



Hermite Specification



Bézier Specification

Bézier Curves

- derivative values of Bezier curve at knots dependent on adjacent points

$$\nabla p_1 = 3(p_2 - p_1)$$
$$\nabla p_4 = 3(p_4 - p_3)$$

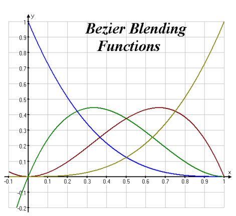
Bézier Blending Functions

- look at blending functions
- family of polynomials called order-3 Bernstein polynomials
  - C(3, k) t^k $(1-t)^{3-k}$; 0<= k <= 3
  - all positive in interval [0,1]
  - sum is equal to 1

$$p(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

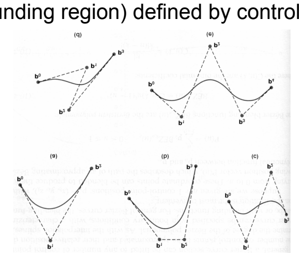
Bézier Blending Functions

- every point on curve is linear combination of control points
- weights of combination are all positive
- sum of weights is 1
- therefore, curve is a convex combination of the control points



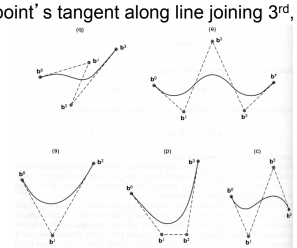
Bézier Curves

- curve will always remain within convex hull (bounding region) defined by control points



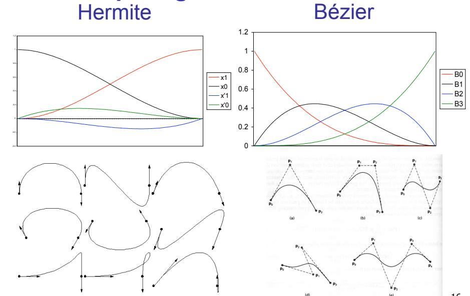
Bézier Curves

- interpolate between first, last control points
- 1st point's tangent along line joining 1st, 2nd pts
- 4th point's tangent along line joining 3rd, 4th pts



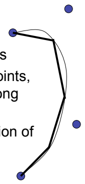
Comparing Hermite and Bézier

Hermite Bézier



Rendering Bezier Curves: Simple

- evaluate curve at fixed set of parameter values, join points with straight lines
- advantage: very simple
- disadvantages:
  - expensive to evaluate the curve at many points
  - no easy way of knowing how fine to sample points, and maybe sampling rate must be different along curve
  - no easy way to adapt: hard to measure deviation of line segment from exact curve

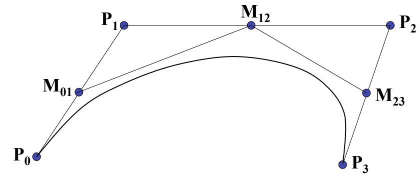


Rendering Beziers: Subdivision

- a cubic Bezier curve can be broken into two shorter cubic Bezier curves that exactly cover original curve
- suggests a rendering algorithm:
  - keep breaking curve into sub-curves
  - stop when control points of each sub-curve are nearly collinear
  - draw the control polygon: polygon formed by control points

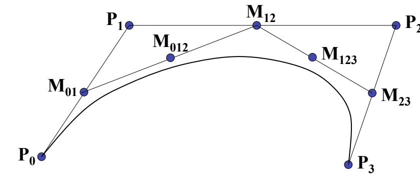
Sub-Dividing Bezier Curves

- step 1: find the midpoints of the lines joining the original control vertices. call them M_{01}, M_{12}, M_{23}



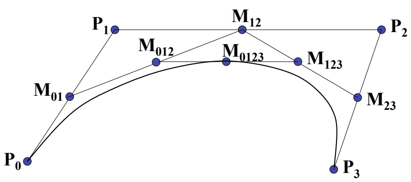
Sub-Dividing Bezier Curves

- step 2: find the midpoints of the lines joining M_{01}, M_{12} and M_{12}, M_{23}. call them M_{012}, M_{123}



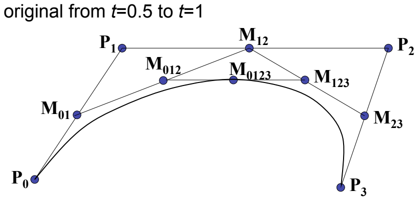
Sub-Dividing Bezier Curves

- step 3: find the midpoint of the line joining M_{012}, M_{123}. call it M_{0123}



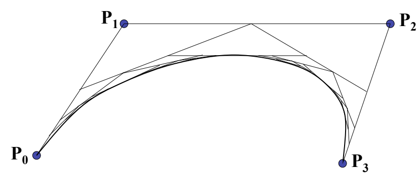
Sub-Dividing Bezier Curves

- curve P_0, M_{01}, M_{012}, M_{0123} exactly follows original from t=0 to t=0.5
- curve M_{0123}, M_{123}, M_{23}, P_3 exactly follows original from t=0.5 to t=1



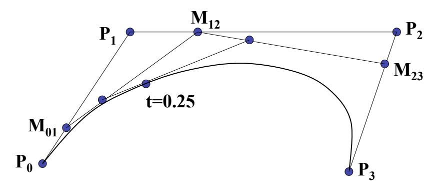
Sub-Dividing Bezier Curves

- continue process to create smooth curve



de Casteljau's Algorithm

- can find the point on a Bezier curve for any parameter value t with similar algorithm
  - for t=0.25, instead of taking midpoints take points 0.25 of the way

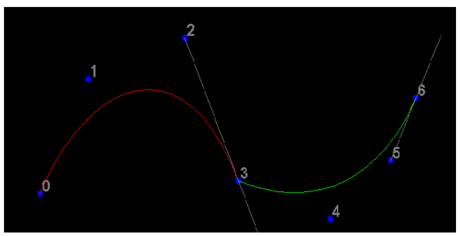


demo: www.saltire.com/applets/advanced_geometry/spline/spline.htm

Longer Curves

- a single cubic Bezier or Hermite curve can only capture a small class of curves
  - at most 2 inflection points
- one solution is to raise the degree
  - allows more control, at the expense of more control points and higher degree polynomials
  - control is not local, one control point influences entire curve
- better solution is to join pieces of cubic curve together into piecewise cubic curves
  - total curve can be broken into pieces, each of which is cubic
  - local control: each control point only influences a limited part of the curve
  - interaction and design is much easier

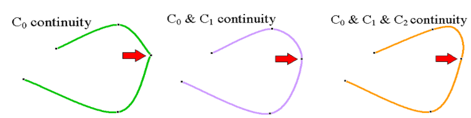
Piecewise Bezier: Continuity Problems



demo: www.cs.princeton.edu/~min/cs426/jar/bezier.html

Continuity

- when two curves joined, typically want some degree of continuity across knot boundary
  - C0, "C-zero", point-wise continuous, curves share same point where they join
  - C1, "C-one", continuous derivatives
  - C2, "C-two", continuous second derivatives



Geometric Continuity

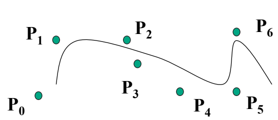
- derivative continuity is important for animation
  - if object moves along curve with constant parametric speed, should be no sudden jump at knots
- for other applications, *tangent continuity* suffices
  - requires that the tangents point in the same direction
  - referred to as G^1 *geometric continuity*
  - curves could be made C^1 with a re-parameterization
  - geometric version of C^2 is G^2, based on curves having the same radius of curvature across the knot

Achieving Continuity

- Hermite curves
  - user specifies derivatives, so C^1 by sharing points and derivatives across knot
- Bezier curves
  - they interpolate endpoints, so C^0 by sharing control pts
  - introduce additional constraints to get C^1
    - parametric derivative is a constant multiple of vector joining first/last 2 control points
    - so C^1 achieved by setting $P_{0,3}=P_{1,0}=J$, and making $P_{0,2}$ and J and $P_{1,1}$ collinear, with $J-P_{0,2}=P_{1,1}-J$
    - C^2 comes from further constraints on $P_{0,1}$ and $P_{1,2}$
  - leads to...

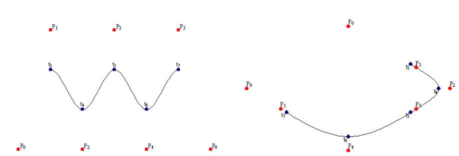
B-Spline Curve

- start with a sequence of control points
- select four from middle of sequence
  (p_{i-2}, p_{i-1}, p_i, p_{i+1})
  - Bezier and Hermite goes between p_{i-2} and p_{i+1}
  - B-Spline doesn't interpolate (touch) any of them but approximates the going through p_{i-1} and p_i



B-Spline

- by far the most popular spline used
- C_0, C_1, and C_2 continuous



demo: www.siggraph.org/education/materials/HyperGraph/modeling/splines/demoprog/curve.html

B-Spline

- locality of points

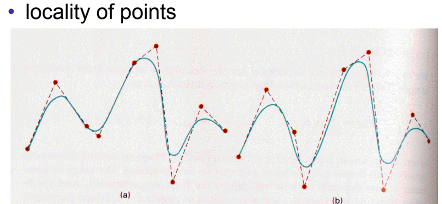


Figure 10-41
Local modification of a B-spline curve. Changing one of the control points in (a) produces curve (b), which is modified only in the neighborhood of the altered control point.