University of British Columbia
CPSC 314 Computer Graphics
Jan-Apr 2008

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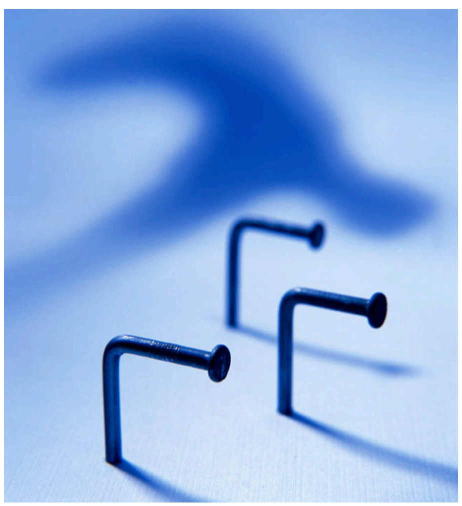
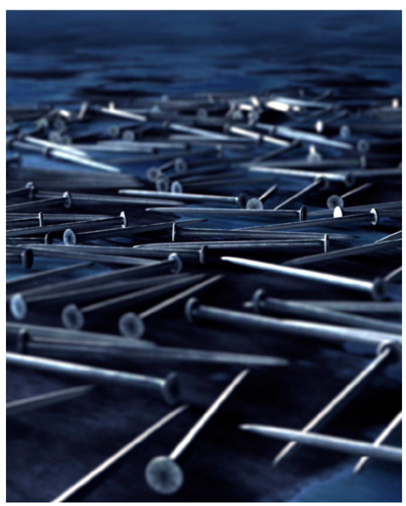
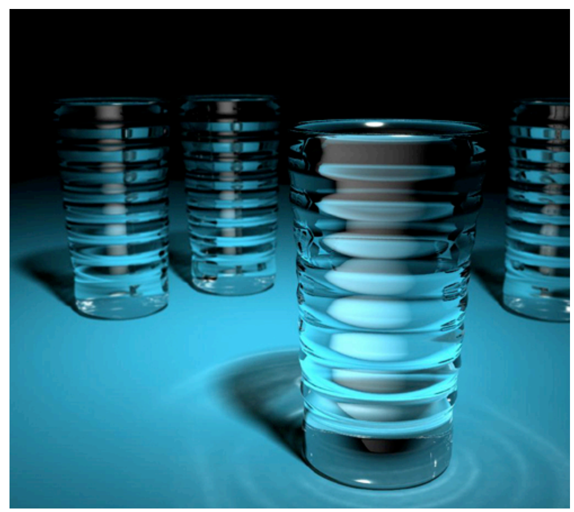
Math Review
Guest Lecturer: Michiel van de Panne

Week 1, Wed Jan 9

http://www.ugrad.cs.ubc.ca/~cs314/Vjan2008

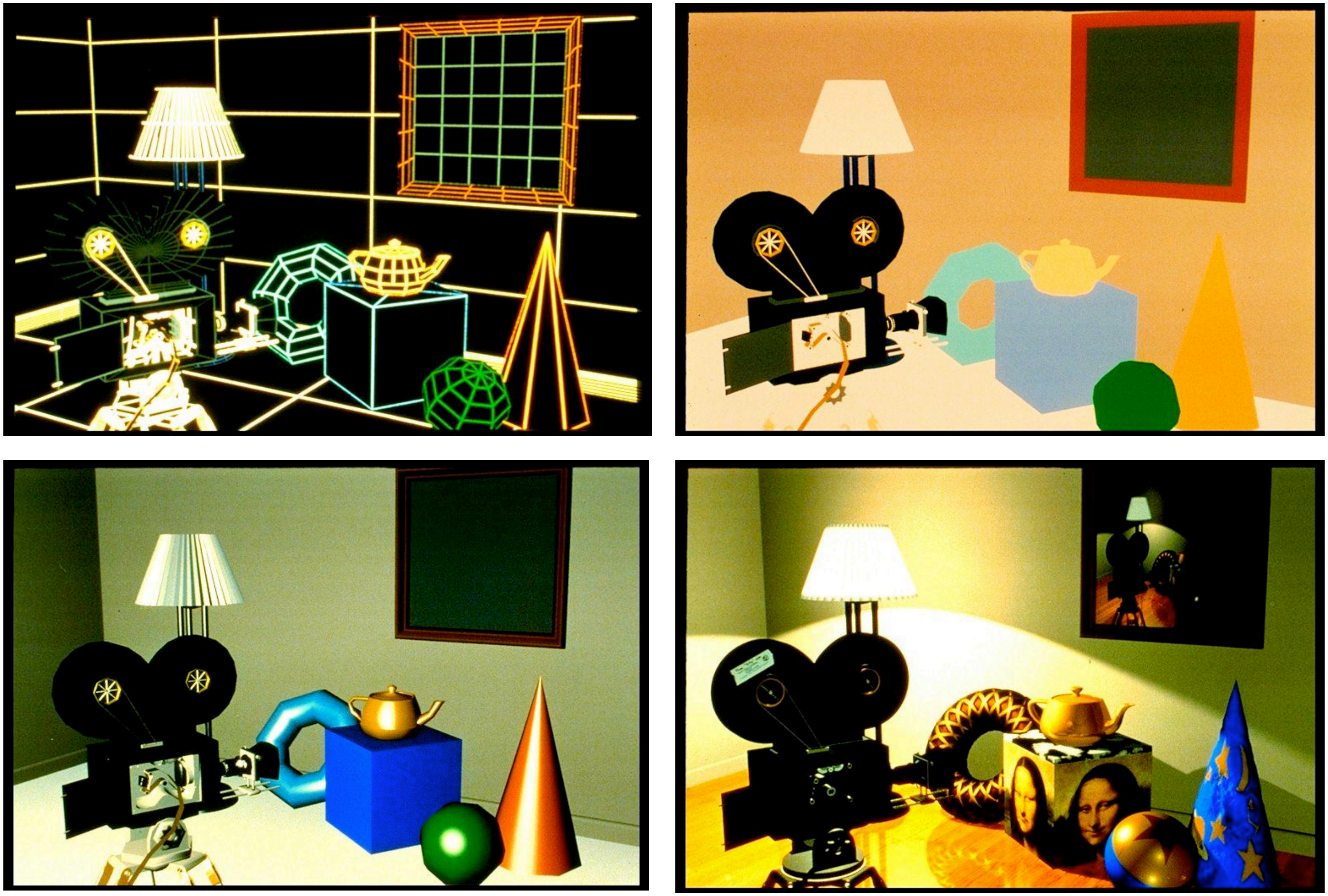
Review: Computer Graphics Defined

- CG uses
  - movies, games, art/design, ads, VR, visualization
- CG state of the art
  - photorealism achievable (in some cases)


http://www.alias.com/eng/etc/fakeorfoto/quiz.html

Review: Rendering Capabilities



www.siggraph.org/education/materials/HyperGraph/shutbug.htm

Readings

- Mon (last time)
  - FCG Chap 1
- Wed (this time)
  - FCG Chap 2
    - except 2.5.1, 2.5.3, 2.7.1, 2.7.3, 2.8, 2.9, 2.11.
  - FCG Chap 5.1-5.2.5
    - except 5.2.3, 5.2.4

Today's Readings

- FCG Chapter 2: Miscellaneous Math
  - skim 2.2 (sets and maps), 2.3 (quadratic eqns)
  - important: 2.3 (trig), 2.4 (vectors), 2.5-6 (lines) 2.10 (linear interpolation)
    - skip 2.5.1, 2.5.3, 2.7.1, 2.7.3, 2.8, 2.9
    - skip 2.11 now (covered later)
- FCG Chapter 5.1-5.25: Linear Algebra
  - skim 5.1 (determinants)
  - important: 5.2.1-5.2.2, 5.2.5 (matrices)
    - skip 5.2.3-4, 5.2.6-7 (matrix numerical analysis)

Notation: Scalars, Vectors, Matrices

- scalar
  - (lower case, italic)
- vector
  - (lower case, bold)
- matrix
  - (upper case, bold)

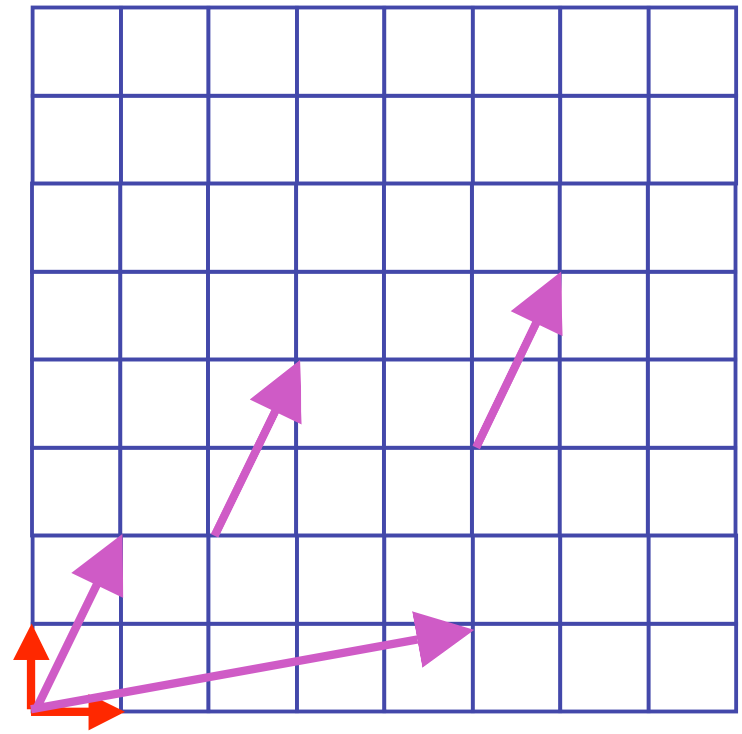
$$a$$

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Vectors

- arrow: length and direction
  - oriented segment in nD space
- offset / displacement
  - location if given origin



Column vs. Row Vectors

- row vectors $\mathbf{a}_{row} = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$
- column vectors

$$\mathbf{a}_{col} = \begin{bmatrix} a_1 \\ a_2 \\ \ldots \\ a_n \end{bmatrix}$$

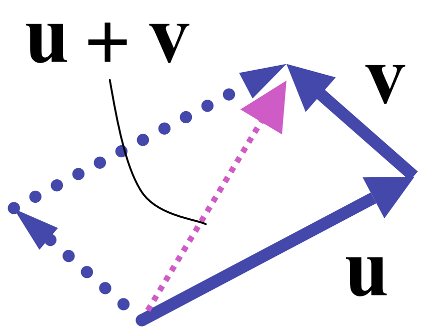
- switch back and forth with transpose

$$\mathbf{a}_{col}^T = \mathbf{a}_{row}$$

Vector-Vector Addition

- add: vector + vector = vector
- parallelogram rule
  - tail to head, complete the triangle

geometric



algebraic

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$$

examples:

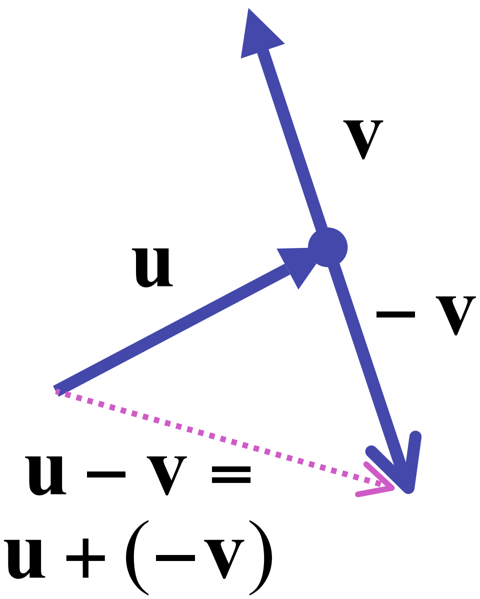
$$(3,2)+(6,4)=(9,6)$$

$$(2,5,1)+(3,1,-1)=(5,6,0)$$

Vector-Vector Subtraction

- subtract: vector - vector = vector

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}$$



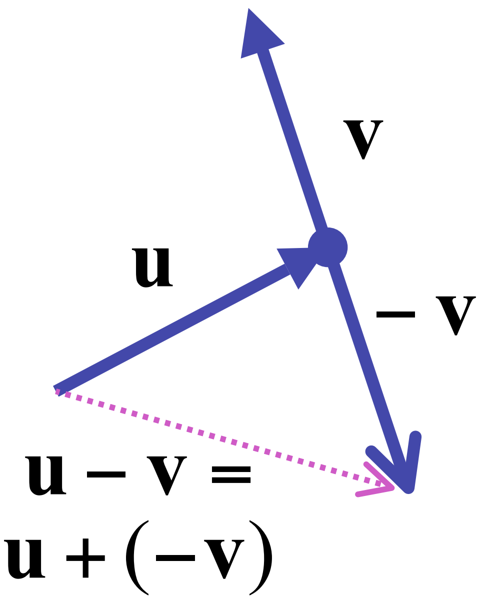
$$(3,2) - (6,4) = (-3,-2)$$

$$(2,5,1) - (3,1,-1) = (-1,4,2)$$

Vector-Vector Subtraction

- subtract: vector - vector = vector

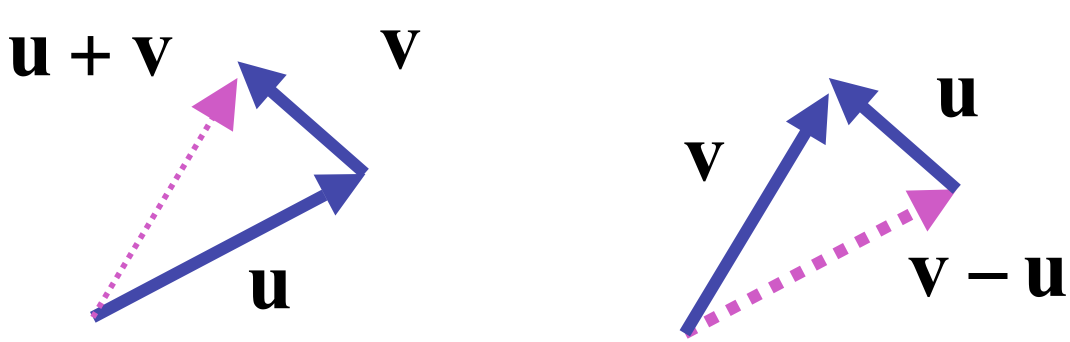
$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}$$



$$(3,2) - (6,4) = (-3,-2)$$

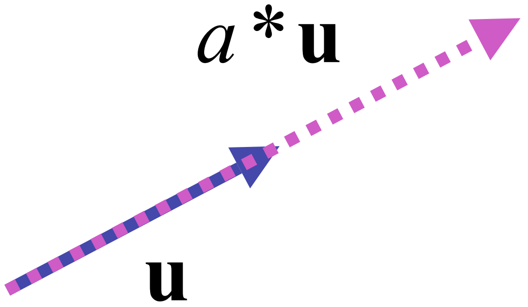
$$(2,5,1) - (3,1,-1) = (-1,4,2)$$

argument reversal



Scalar-Vector Multiplication

- multiply: scalar * vector = vector
  - vector is scaled



$$a * \mathbf{u} = (a * u_1, a * u_2, a * u_3)$$

$$2 * (3,2) = (6,4)$$

$$.5 * (2,5,1) = (1,2.5,.5)$$

Vector-Vector Multiplication

- multiply: vector * vector = scalar
- dot product, aka inner product

$$\mathbf{u} \bullet \mathbf{v}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1 * v_1) + (u_2 * v_2) + (u_3 * v_3)$$

Vector-Vector Multiplication

- multiply: vector * vector = scalar
- dot product, aka inner product

$$\mathbf{u} \bullet \mathbf{v}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1 * v_1) + (u_2 * v_2) + (u_3 * v_3)$$

Vector-Vector Multiplication

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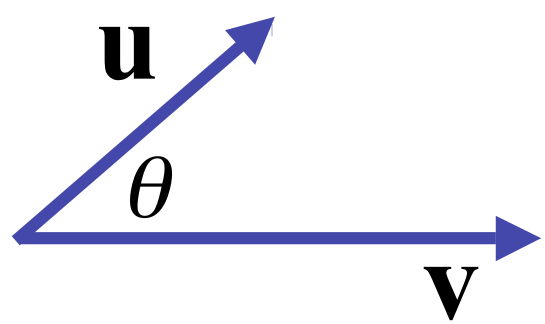
$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$$

- geometric interpretation
  - lengths, angles
  - can find angle between two vectors

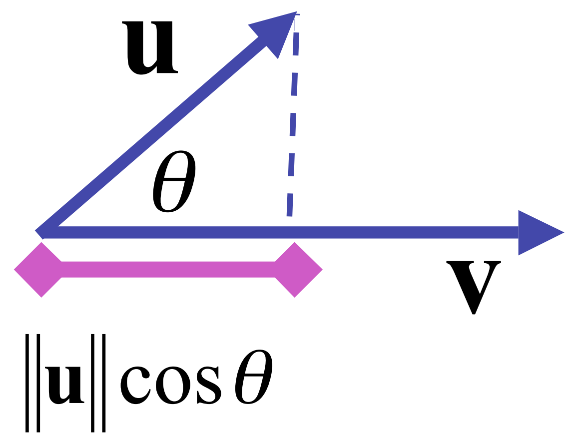


Dot Product Geometry

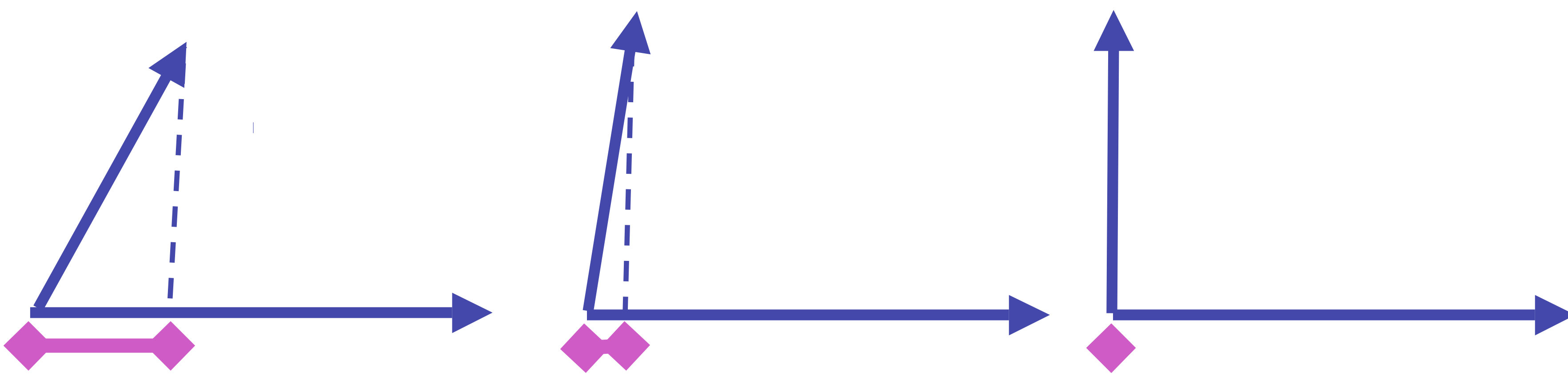
- can find length of projection of u onto v

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

$$\|\mathbf{u}\|\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|}$$



- as lines become perpendicular, $\mathbf{u} \bullet \mathbf{v} \rightarrow 0$



Dot Product Example

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \left(u_1 * v_1\right) + \left(u_2 * v_2\right) + \left(u_3 * v_3\right)$$

$$\begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} = (6 * 1) + (1 * 7) + (2 * 3) = 6 + 7 + 6 = 19$$

Vector-Vector Multiplication, The Sequel

- multiply: vector * vector = vector
- cross product
  - algebraic

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Vector-Vector Multiplication, The Sequel

- multiply: vector * vector = vector
- cross product
  - algebraic

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Vector-Vector Multiplication, The Sequel

- multiply: vector * vector = vector
- cross product
  - algebraic

$$\begin{matrix}3\\1\\2\end{matrix}\quad\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\times\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}$$

Vector-Vector Multiplication, The Sequel

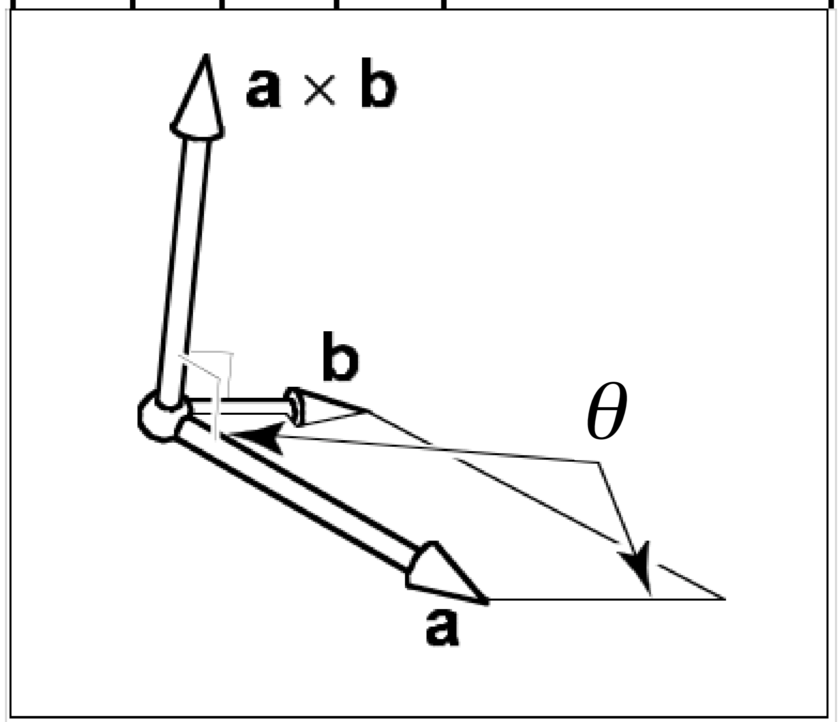
- multiply: vector * vector = vector
- cross product
  - algebraic

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

  - geometric

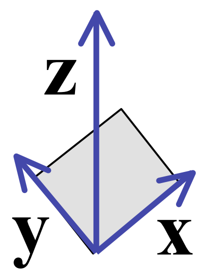
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$$

    - $\|\mathbf{a} \times \mathbf{b}\|$ parallelogram area
    - $\mathbf{a} \times \mathbf{b}$ perpendicular to parallelogram



RHS vs. LHS Coordinate Systems

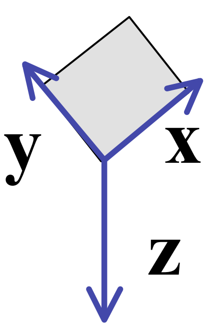
- right-handed coordinate system convention



right hand rule:
index finger x, second finger y;
right thumb points up

$$\mathbf{z} = \mathbf{x} \times \mathbf{y}$$

- left-handed coordinate system



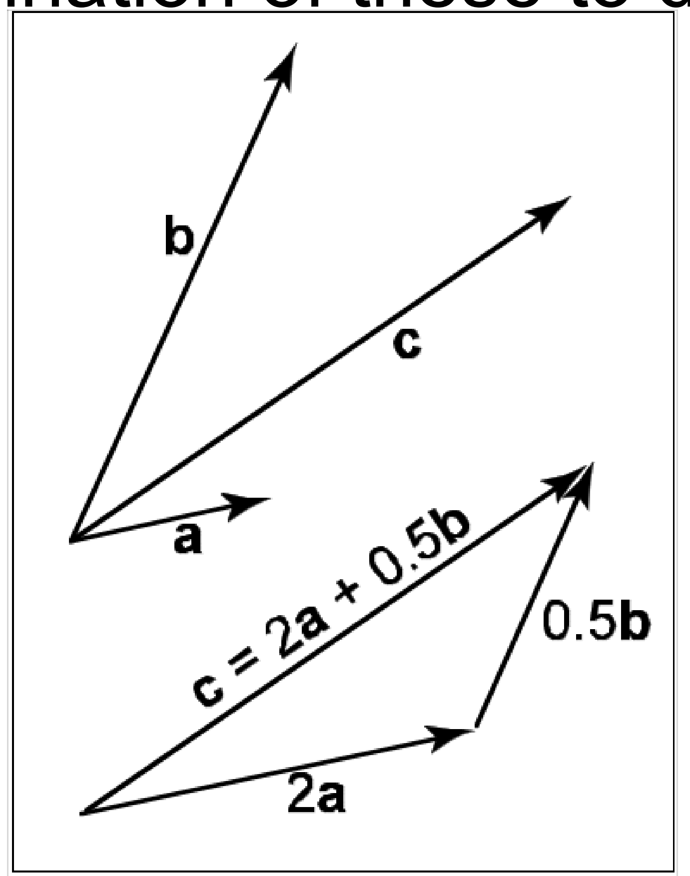
left hand rule:
index finger x, second finger y;
left thumb points down

$$\mathbf{z} = \mathbf{x} \times \mathbf{y}$$

Basis Vectors

- take any two vectors that are linearly independent (nonzero and nonparallel)
  - can use linear combination of these to define any other vector:

$$\mathbf{c} = w_1\mathbf{a} + w_2\mathbf{b}$$



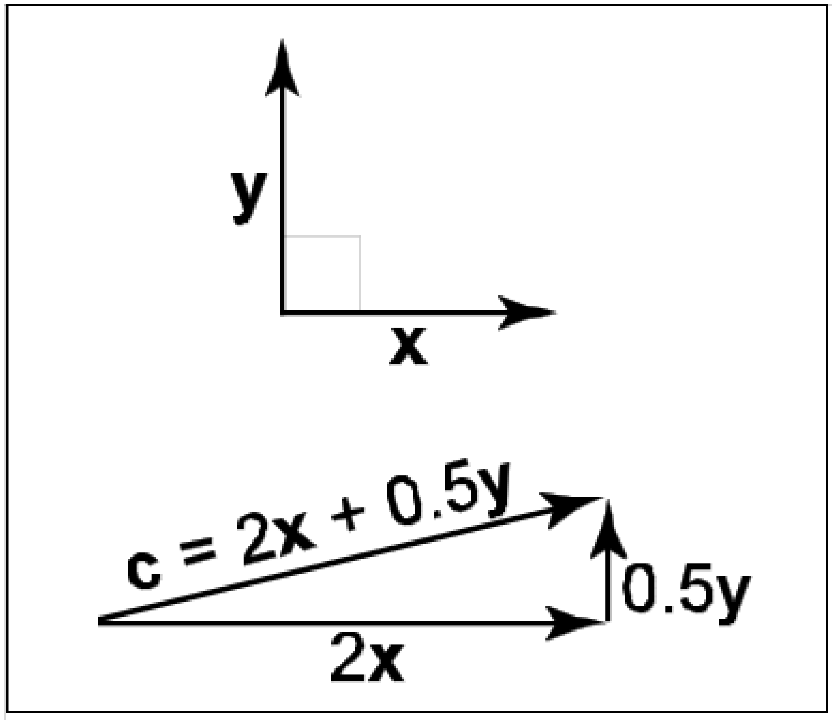
Orthonormal Basis Vectors

- if basis vectors are orthonormal (orthogonal (mutually perpendicular) and unit length)
  - we have Cartesian coordinate system
  - familiar Pythagorean definition of distance

orthonormal algebraic properties

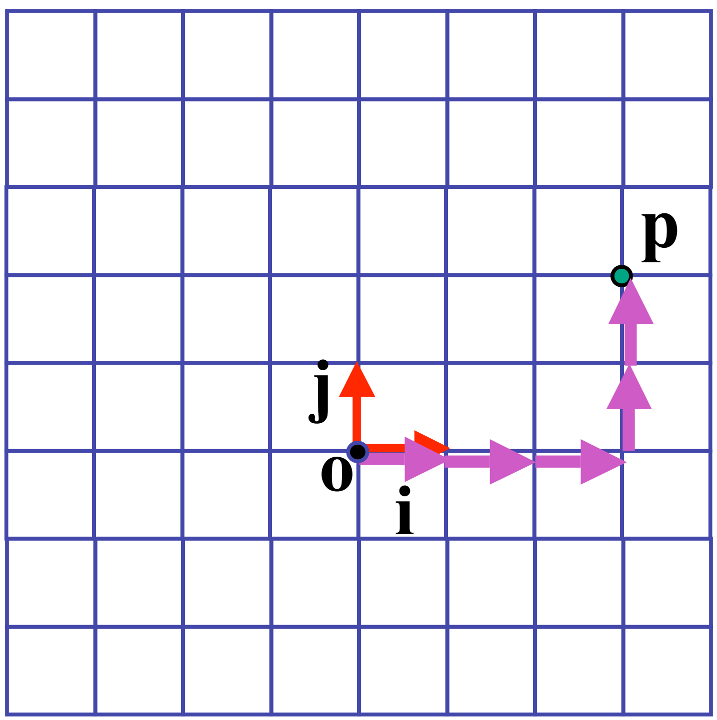
$$\|\mathbf{x}\| = \|\mathbf{y}\| = 1,$$

$$\mathbf{x} \bullet \mathbf{y} = 0$$



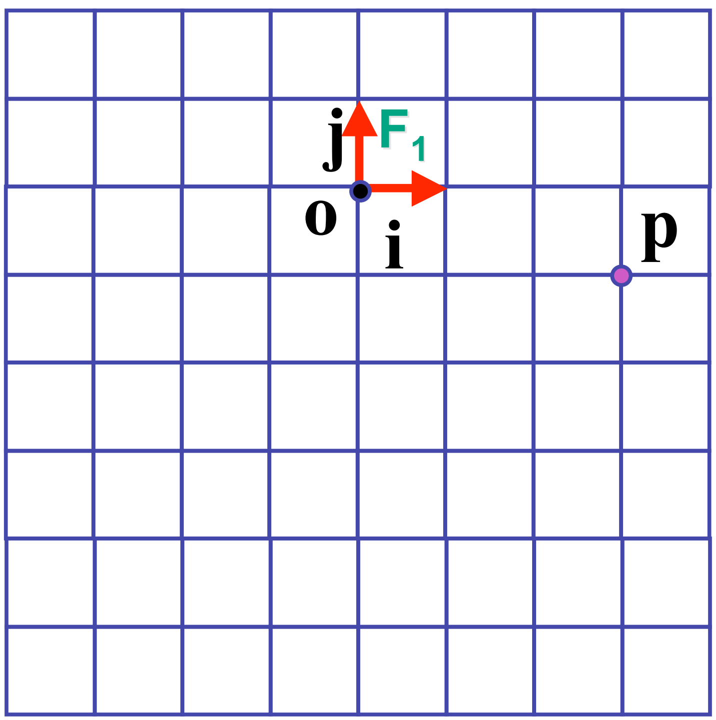
Basis Vectors and Origins

- coordinate system: just basis vectors
  - can only specify offset: vectors
- coordinate frame: basis vectors and origin
  - can specify location as well as offset: points



$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

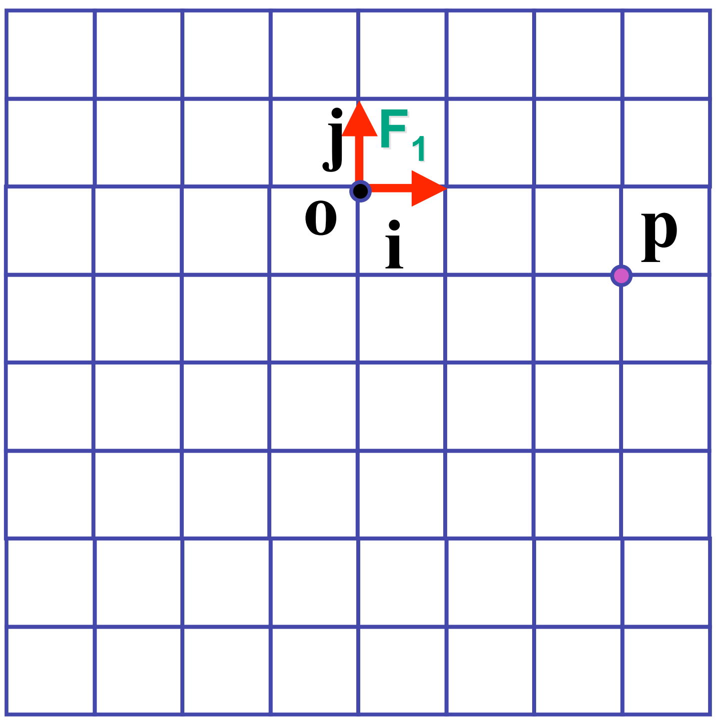
Working with Frames



$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

F_1

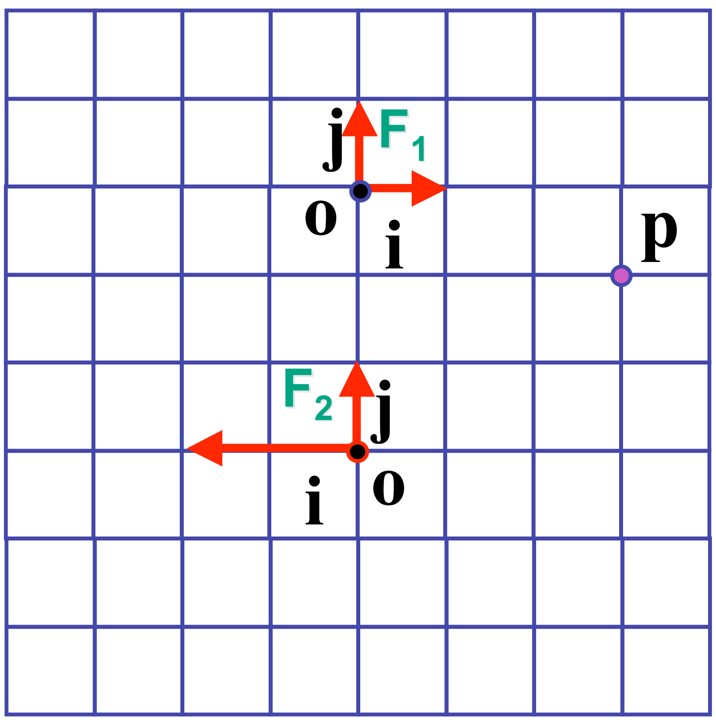
Working with Frames



$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

F_1 **p = (3,-1)**

Working with Frames

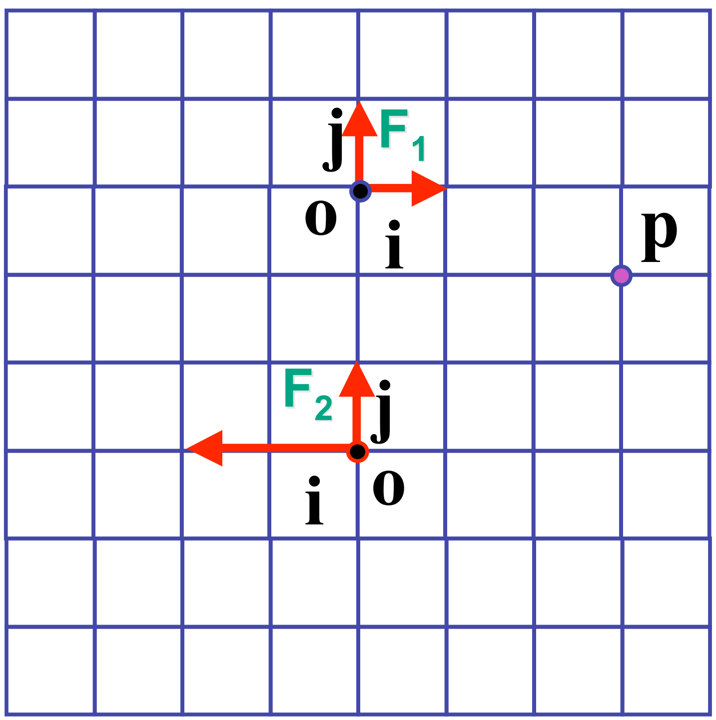


$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

F_1 p = (3,-1)

F_2

Working with Frames

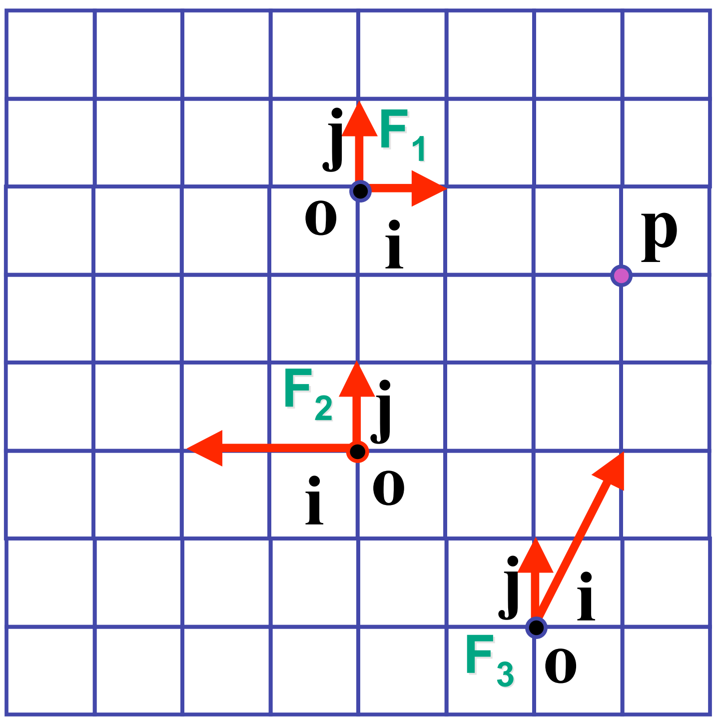


$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

F_1 p = (3,-1)

F_2 p = (-1.5,2)

Working with Frames



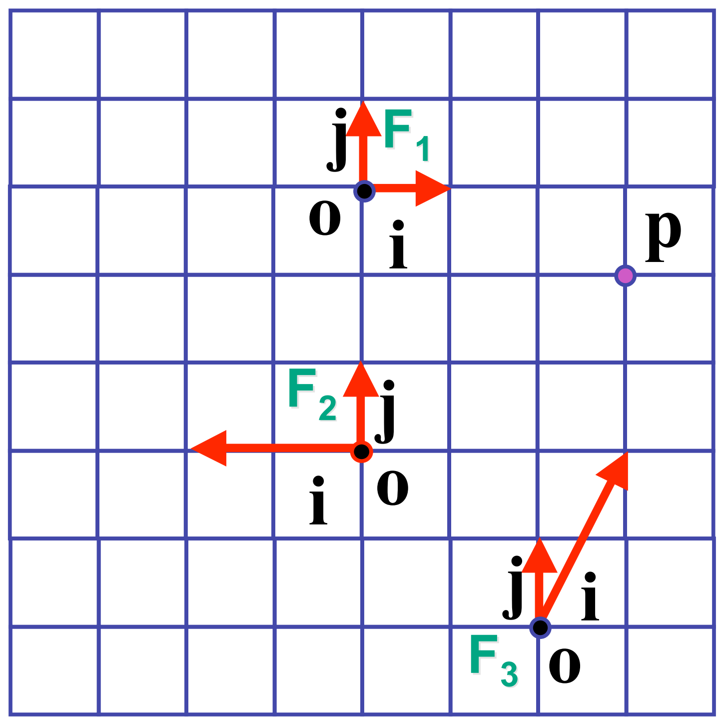
$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

F_1 p = (3,-1)

F_2 p = (-1.5,2)

F_3

Working with Frames



$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

F_1 p = (3,-1)

F_2 p = (-1.5,2)

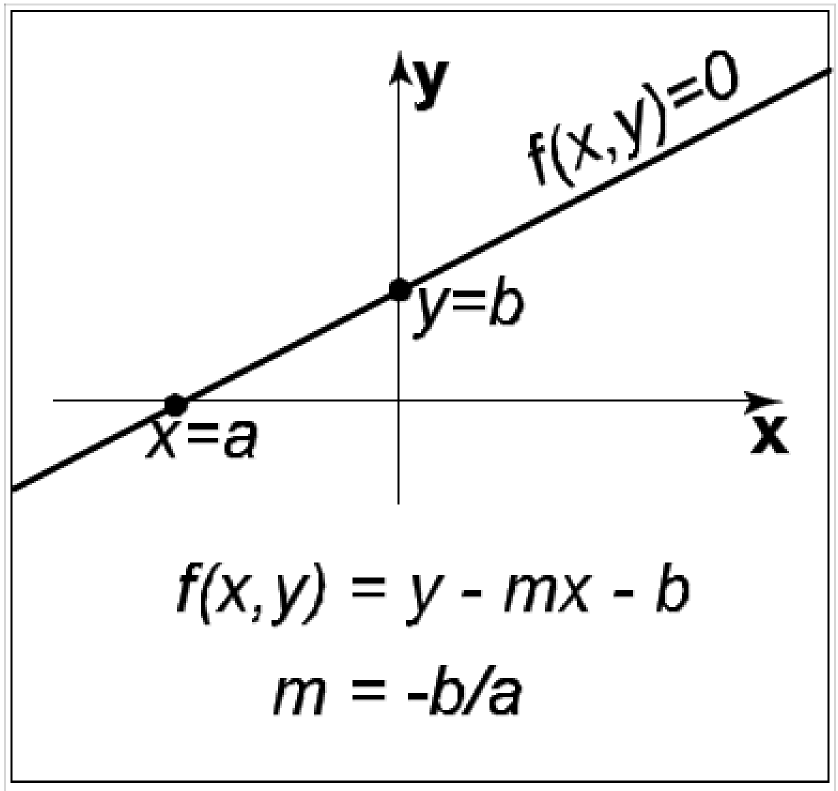
F_3 p = (1,2)

Named Coordinate Frames

- origin and basis vectors $\mathbf{p} = \mathbf{o} + a\mathbf{x} + b\mathbf{y} + c\mathbf{z}$
- pick canonical frame of reference
  - then don't have to store origin, basis vectors
  - just $\mathbf{p} = (a, b, c)$
  - convention: Cartesian orthonormal one on previous slide
- handy to specify others as needed
  - airplane nose, looking over your shoulder, ...
  - really common ones given names in CG
    - object, world, camera, screen, ...

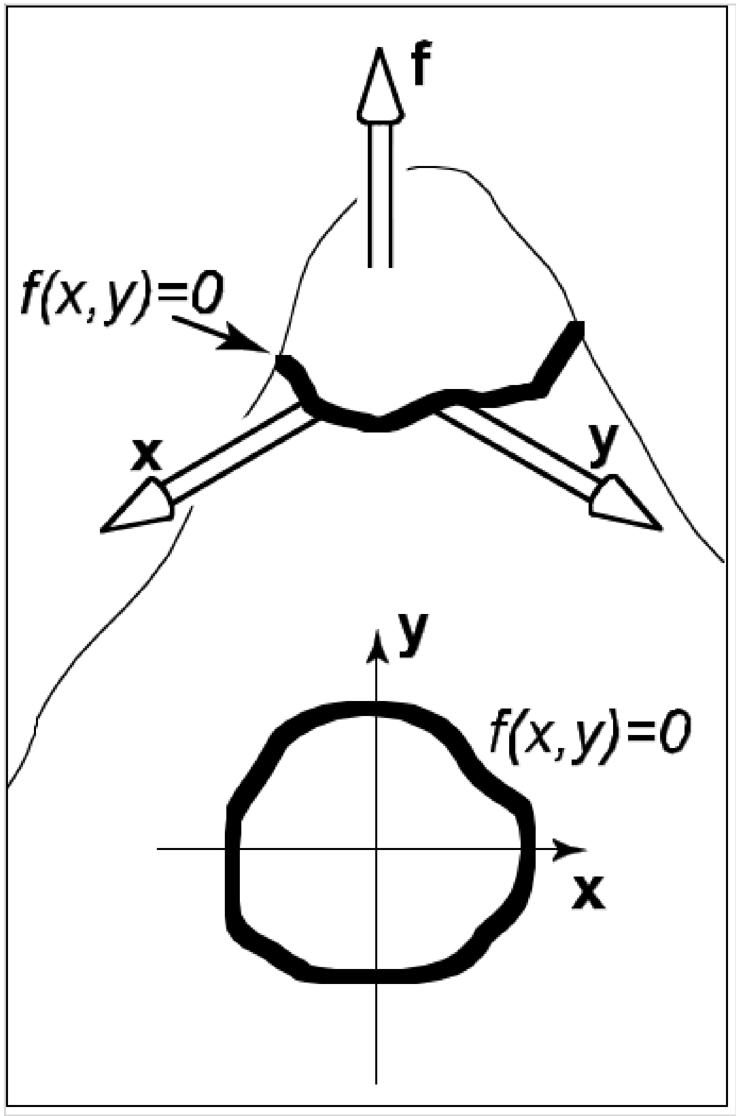
Lines

- slope-intercept form
  - $y = mx + b$
- implicit form
  - $y - mx - b = 0$
  - $Ax + By + C = 0$
  - $f(x,y) = 0$



Implicit Functions

- find where function is 0
  - plug in (x,y), check if
    - 0: on line
    - < 0: inside
    - > 0: outside
- analogy: terrain
  - sea level: f=0
  - altitude: function value
  - topo map: equal-value contours (level sets)



Implicit Circles

- $f(x,y) = (x - x_c)^2 + (y - y_c)^2 - r^2$
  - circle is points (x,y) where f(x,y) = 0
- $p = (x,y), c = (x_c, y_c) : (\mathbf{p} - \mathbf{c}) \bullet (\mathbf{p} - \mathbf{c}) - r^2 = 0$
  - points **p** on circle have property that vector from **c** to **p** dotted with itself has value r^2
- $\|\mathbf{p} - \mathbf{c}\|^2 - r^2 = 0$
  - points points **p** on the circle have property that squared distance from **c** to **p** is r^2
- $\|\mathbf{p} - \mathbf{c}\| - r = 0$
  - points **p** on circle are those a distance *r* from center point **c**

Parametric Curves

- parameter: index that changes continuously
  - (x,y): point on curve
  - t: parameter

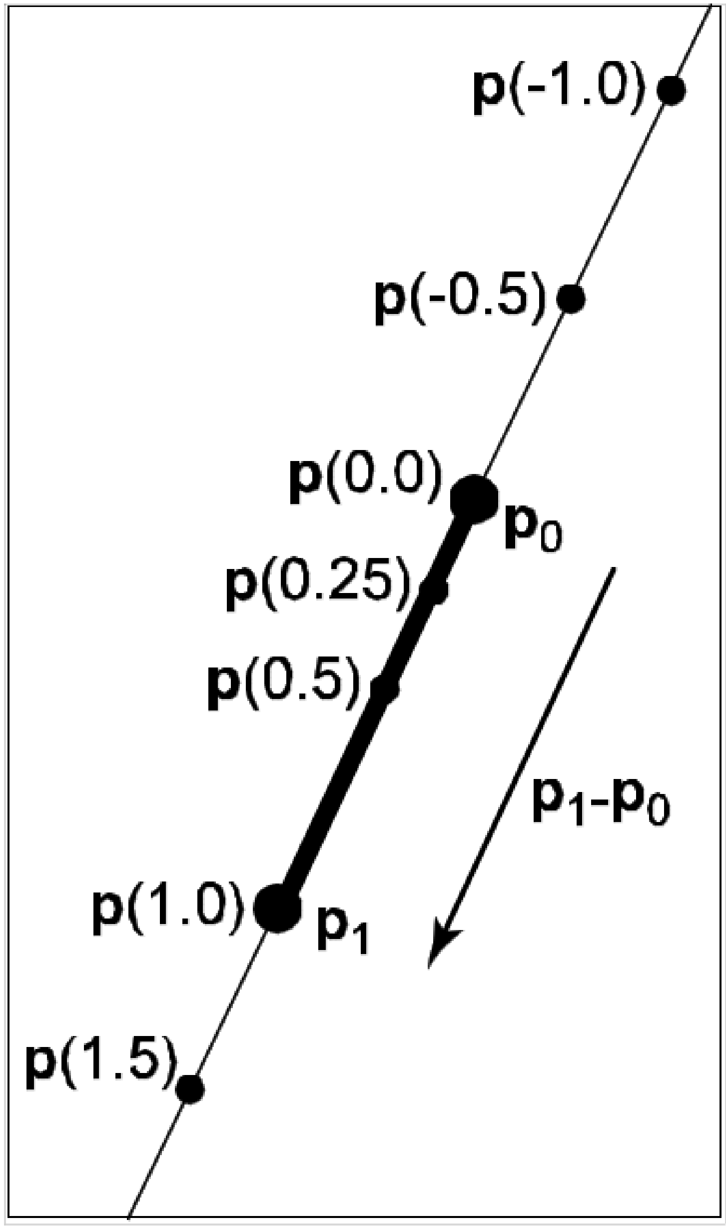
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g(t) \\ h(t) \end{bmatrix}$$

- vector form
  - $\mathbf{p} = f(t)$

2D Parametric Lines

- $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 + t(x_1 - x_0) \\ y_0 + t(y_1 - y_0) \end{bmatrix}$$
- $\mathbf{p}(t) = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0)$
- $\mathbf{p}(t) = \mathbf{o} + t(\mathbf{d})$

- start at point $\mathbf{p}_0$,
  go towards $\mathbf{p}_1$,
  according to parameter t
  - $\mathbf{p}(0) = \mathbf{p}_0$, $\mathbf{p}(1) = \mathbf{p}_1$



Linear Interpolation

- parametric line is example of general concept
  - $\mathbf{p}(t) = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0)$
  - interpolation
    - **p** goes through **a** at *t* = 0
    - **p** goes through **b** at *t* = 1
  - linear
    - weights *t*, (1-*t*) are linear polynomials in *t*

Matrix-Matrix Addition

- add: matrix + matrix = matrix

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} n_{11} + m_{11} & n_{12} + m_{12} \\ n_{21} + m_{21} & n_{22} + m_{22} \end{bmatrix}$$

- example

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 1 + (-2) & 3 + 5 \\ 2 + 7 & 4 + 1 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 9 & 5 \end{bmatrix}$$

Scalar-Matrix Multiplication

- multiply: scalar * matrix = matrix

$$a\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} a*m_{11} & a*m_{12} \\ a*m_{21} & a*m_{22} \end{bmatrix}$$

- example

$$3\begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3*2 & 3*4 \\ 3*1 & 3*5 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 3 & 15 \end{bmatrix}$$

Matrix-Matrix Multiplication

- can only multiply (n,k) by (k,m): number of left cols = number of right rows
  - legal

$$\begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix} \begin{bmatrix} h & i \\ j & k \\ l & m \end{bmatrix}$$

  - undefined

$$\begin{bmatrix} a & b & c \\ e & f & g \\ o & p & q \end{bmatrix} \begin{bmatrix} h & i \\ j & k \end{bmatrix}$$

Matrix-Matrix Multiplication

- row by column

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

Matrix-Matrix Multiplication

- row by column

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

$$p_{21} = m_{21}n_{11} + m_{22}n_{21}$$

Matrix-Matrix Multiplication

- row by column

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

$$p_{21} = m_{21}n_{11} + m_{22}n_{21}$$

$$p_{12} = m_{11}n_{12} + m_{12}n_{22}$$

Matrix-Matrix Multiplication

- row by column

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

$$p_{21} = m_{21}n_{11} + m_{22}n_{21}$$

$$p_{12} = m_{11}n_{12} + m_{12}n_{22}$$

$$p_{22} = m_{21}n_{12} + m_{22}n_{22}$$

Matrix-Matrix Multiplication

- row by column

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

$$p_{21} = m_{21}n_{11} + m_{22}n_{21}$$

$$p_{12} = m_{11}n_{12} + m_{12}n_{22}$$

$$p_{22} = m_{21}n_{12} + m_{22}n_{22}$$

- noncommutative: **AB != BA**

Matrix-Vector Multiplication

- points as column vectors: postmultiply

$$\begin{bmatrix} x' \\ y' \\ z' \\ h' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ h \end{bmatrix} \qquad \mathbf{p}' = \mathbf{M}\mathbf{p}$$

- points as row vectors: premultiply

$$\begin{bmatrix} x' & y' & z' & h' \end{bmatrix} = \begin{bmatrix} x & y & z & h \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}^T \qquad \mathbf{p}'^T = \mathbf{p}^T \mathbf{M}^T$$

Matrices

- transpose $$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}^T = \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix}$$
- identity $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
- inverse $$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$
  - not all matrices are invertible

Matrices and Linear Systems

- linear system of n equations, n unknowns

$$3x + 7y + 2z = 4$$
$$2x - 4y - 3z = -1$$
$$5x + 2y + z = 1$$

- matrix form **Ax=b**

$$\begin{bmatrix} 3 & 7 & 2 \\ 2 & -4 & -3 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$