



Tamara Munzner

### Math Review

Guest Lecturer: Michiel van de Panne

Week 1, Wed Jan 9

<http://www.ugrad.cs.ubc.ca/~cs314/vjan2008>

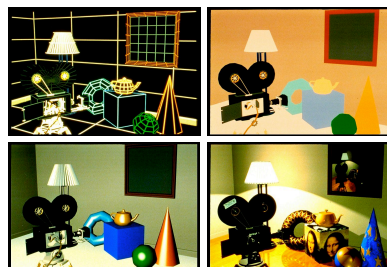
### Review: Computer Graphics Defined

- CG uses
  - movies, games, art/design, ads, VR, visualization
- CG state of the art
  - photorealism achievable (in some cases)



<http://www.alias.com/eng/etc/fakeorfoto/quiz.html>

### Review: Rendering Capabilities



[www.siggraph.org/education/materials/HyperGraph/shutbug.htm](http://www.siggraph.org/education/materials/HyperGraph/shutbug.htm)

### Readings

- Mon (last time)
  - FCG Chap 1
- Wed (this time)
  - FCG Chap 2
    - except 2.5.1, 2.5.3, 2.7.1, 2.7.3, 2.8, 2.9, 2.11.
  - FCG Chap 5.1-5.2.5
    - except 5.2.3, 5.2.4

### Today's Readings

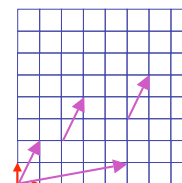
- FCG Chapter 2: Miscellaneous Math
  - skim 2.2 (sets and maps), 2.3 (quadratic eqns)
  - important: 2.3 (trig), 2.4 (vectors), 2.5-6 (lines) 2.10 (linear interpolation)
    - skip 2.5.1, 2.5.3, 2.7.1, 2.7.3, 2.8, 2.9
    - skip 2.11 now (covered later)
- FCG Chapter 5.1-5.25: Linear Algebra
  - skim 5.1 (determinants)
  - important: 5.2.1-5.2.2, 5.2.5 (matrices)
    - skip 5.2.3-4, 5.2.6-7 (matrix numerical analysis)

### Notation: Scalars, Vectors, Matrices

- scalar  $a$ 
  - (lower case, italic)
- vector  $\mathbf{a} = [a_1 \ a_2 \ \dots \ a_n]$ 
  - (lower case, bold)
- matrix  $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ 
  - (upper case, bold)

### Vectors

- arrow: length and direction
- oriented segment in nD space
- offset / displacement
  - location if given origin



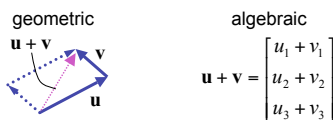
### Column vs. Row Vectors

- row vectors  $\mathbf{a}_{row} = [a_1 \ a_2 \ \dots \ a_n]$
- column vectors  $\mathbf{a}_{col} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix}$
- switch back and forth with transpose

$$\mathbf{a}_{col}^T = \mathbf{a}_{row}$$

### Vector-Vector Addition

- add: vector + vector = vector
- parallelogram rule
  - tail to head, complete the triangle



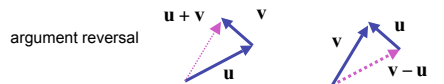
examples:  $(3,2) + (6,4) = (9,6)$   
 $(2,5,1) + (3,1,-1) = (5,6,0)$

### Vector-Vector Subtraction

- subtract: vector - vector = vector
- $\mathbf{u} - \mathbf{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}$
- $(3,2) - (6,4) = (-3,-2)$   
 $(2,5,1) - (3,1,-1) = (-1,4,2)$
- 

### Vector-Vector Subtraction

- subtract: vector - vector = vector
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- $(3,2) - (6,4) = (-3,-2)$   
 $(2,5,1) - (3,1,-1) = (-1,4,2)$
- 



### Scalar-Vector Multiplication

- multiply: scalar \* vector = vector
- vector is scaled

$a * \mathbf{u} = (a * u_1, a * u_2, a * u_3)$

$2 * (3,2) = (6,4)$   
 $.5 * (2,5,1) = (1,2.5,.5)$

### Vector-Vector Multiplication

- multiply: vector \* vector = scalar
- dot product, aka inner product  $\mathbf{u} \cdot \mathbf{v}$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1 * v_1) + (u_2 * v_2) + (u_3 * v_3)$$

### Vector-Vector Multiplication

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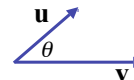
### Vector-Vector Multiplication

- multiply: vector \* vector = scalar
- dot product, aka inner product  $\mathbf{u} \cdot \mathbf{v}$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1 * v_1) + (u_2 * v_2) + (u_3 * v_3)$$

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

- geometric interpretation
  - lengths, angles
  - can find angle between two vectors



### Dot Product Geometry

- can find length of projection of u onto v

$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$

$\|\mathbf{u}\| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|}$

- as lines become perpendicular,  $\mathbf{u} \cdot \mathbf{v} \rightarrow 0$



## Dot Product Example

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_1 * v_1) + (u_2 * v_2) + (u_3 * v_3)$$

$$\begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} = (6*1) + (1*7) + (2*3) = 6 + 7 + 6 = 19$$

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## Vector-Vector Multiplication, The Sequel

- multiply: vector \* vector = vector
- cross product
  - algebraic

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

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## Vector-Vector Multiplication, The Sequel

- multiply: vector \* vector = vector
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$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

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## Vector-Vector Multiplication, The Sequel

- multiply: vector \* vector = vector
- cross product
  - algebraic

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

blah blah

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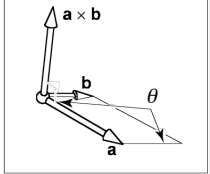
## Vector-Vector Multiplication, The Sequel

- multiply: vector \* vector = vector
- cross product
  - algebraic
  - geometric

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$$

- $\|\mathbf{a} \times \mathbf{b}\|$  parallelogram area
- $\mathbf{a} \times \mathbf{b}$  perpendicular to parallelogram



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## RHS vs. LHS Coordinate Systems

- right-handed coordinate system **convention**
- left-handed coordinate system



right hand rule:  
index finger x, second finger y;  
right thumb points up  
 $\mathbf{z} = \mathbf{x} \times \mathbf{y}$



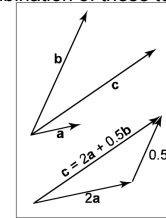
left hand rule:  
index finger x, second finger y;  
left thumb points down  
 $\mathbf{z} = \mathbf{x} \times \mathbf{y}$

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## Basis Vectors

- take any two vectors that are **linearly independent** (nonzero and nonparallel)
  - can use linear combination of these to define any other vector:

$$\mathbf{c} = w_1 \mathbf{a} + w_2 \mathbf{b}$$



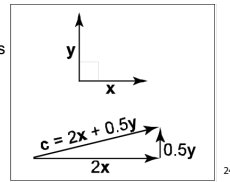
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## Orthonormal Basis Vectors

- if basis vectors are **orthonormal** (orthogonal and unit length)
  - we have Cartesian coordinate system
  - familiar Pythagorean definition of distance

orthonormal algebraic properties

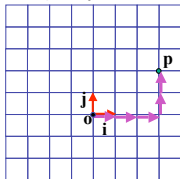
$$\|\mathbf{x}\| = \|\mathbf{y}\| = 1, \quad \mathbf{x} \cdot \mathbf{y} = 0$$



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## Basis Vectors and Origins

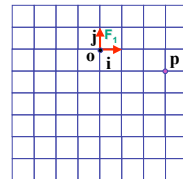
- coordinate system**: just basis vectors
  - can only specify offset: vectors
- coordinate frame**: basis vectors and origin
  - can specify location as well as offset: points



$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

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## Working with Frames

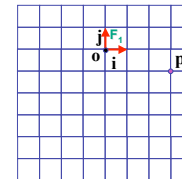


$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$\mathbf{F}_1$

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## Working with Frames

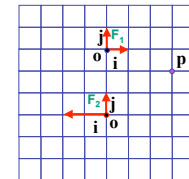


$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$\mathbf{F}_1 \quad \mathbf{p} = (3, -1)$

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## Working with Frames



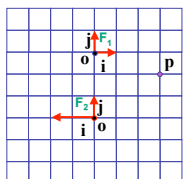
$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$\mathbf{F}_1 \quad \mathbf{p} = (3, -1)$

$\mathbf{F}_2$

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## Working with Frames



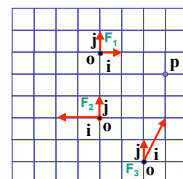
$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$\mathbf{F}_1 \quad \mathbf{p} = (3, -1)$

$\mathbf{F}_2 \quad \mathbf{p} = (-1.5, 2)$

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## Working with Frames



$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

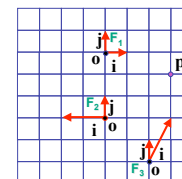
$\mathbf{F}_1 \quad \mathbf{p} = (3, -1)$

$\mathbf{F}_2 \quad \mathbf{p} = (-1.5, 2)$

$\mathbf{F}_3$

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## Working with Frames



$$\mathbf{p} = \mathbf{o} + x\mathbf{i} + y\mathbf{j}$$

$\mathbf{F}_1 \quad \mathbf{p} = (3, -1)$

$\mathbf{F}_2 \quad \mathbf{p} = (-1.5, 2)$

$\mathbf{F}_3 \quad \mathbf{p} = (1, 2)$

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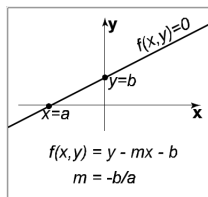
## Named Coordinate Frames

- origin and basis vectors  $\mathbf{p} = \mathbf{o} + ax + by + cz$
- pick canonical frame of reference
  - then don't have to store origin, basis vectors
  - just  $\mathbf{p} = (a, b, c)$
  - convention: Cartesian orthonormal one on previous slide
- handy to specify others as needed
  - airplane nose, looking over your shoulder, ...
  - really common ones given names in CG
    - object, world, camera, screen, ...

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## Lines

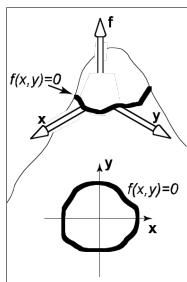
- slope-intercept form
  - $y = mx + b$
- implicit form
  - $y - mx - b = 0$
  - $Ax + By + C = 0$
  - $f(x,y) = 0$



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## Implicit Functions

- find where function is 0
  - plug in  $(x,y)$ , check if
    - 0: on line
    - < 0: inside
    - > 0: outside
- analogy: terrain
  - sea level:  $f=0$
  - altitude: function value
  - topo map: equal-value contours (level sets)



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## Implicit Circles

- $f(x,y) = (x-x_c)^2 + (y-y_c)^2 - r^2$ 
  - circle is points  $(x,y)$  where  $f(x,y) = 0$
- $p = (x,y), c = (x_c, y_c): (p-c) \cdot (p-c) - r^2 = 0$ 
  - points  $p$  on circle have property that vector from  $c$  to  $p$  dotted with itself has value  $r^2$
- $\|p-c\|^2 - r^2 = 0$ 
  - points  $p$  on the circle have property that squared distance from  $c$  to  $p$  is  $r^2$
- $\|p-c\| - r = 0$ 
  - points  $p$  on circle are those a distance  $r$  from center point  $c$

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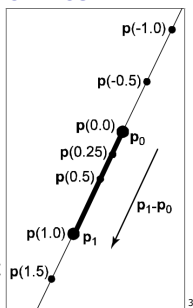
## Parametric Curves

- parameter: index that changes continuously
  - $(x,y)$ : point on curve
  - $t$ : parameter
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g(t) \\ h(t) \end{bmatrix}$$
- vector form
  - $p = f(t)$

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## 2D Parametric Lines

- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 + t(x_1 - x_0) \\ y_0 + t(y_1 - y_0) \end{bmatrix}$
- $p(t) = p_0 + t(p_1 - p_0)$
- $p(t) = o + t(d)$
- start at point  $p_0$ , go towards  $p_1$ , according to parameter  $t$ 
  - $p(0) = p_0, p(1) = p_1$



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## Linear Interpolation

- parametric line is example of general concept
  - $p(t) = p_0 + t(p_1 - p_0)$
  - interpolation
    - $p$  goes through  $a$  at  $t = 0$
    - $p$  goes through  $b$  at  $t = 1$
  - linear
    - weights  $t, (1-t)$  are linear polynomials in  $t$

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## Matrix-Matrix Addition

- add: matrix + matrix = matrix

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} n_{11} + m_{11} & n_{12} + m_{12} \\ n_{21} + m_{21} & n_{22} + m_{22} \end{bmatrix}$$

- example

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 1 + (-2) & 3 + 5 \\ 2 + 7 & 4 + 1 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 9 & 5 \end{bmatrix}$$

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## Scalar-Matrix Multiplication

- multiply: scalar \* matrix = matrix

$$a \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} a * m_{11} & a * m_{12} \\ a * m_{21} & a * m_{22} \end{bmatrix}$$

- example

$$3 \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 * 2 & 3 * 4 \\ 3 * 1 & 3 * 5 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 3 & 15 \end{bmatrix}$$

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## Matrix-Matrix Multiplication

- can only multiply  $(n,k)$  by  $(k,m)$ : number of left cols = number of right rows

- legal

$$\begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix} \begin{bmatrix} h & i \\ j & k \\ l & m \end{bmatrix}$$

- undefined

$$\begin{bmatrix} a & b & c \\ e & f & g \\ o & p & q \end{bmatrix} \begin{bmatrix} h & i \\ j & k \end{bmatrix}$$

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## Matrix-Matrix Multiplication

- row by column

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

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## Matrix-Matrix Multiplication

- row by column

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

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$$p_{21} = m_{21}n_{11} + m_{22}n_{21}$$

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## Matrix-Matrix Multiplication

- row by column

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$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

$$p_{21} = m_{21}n_{11} + m_{22}n_{21}$$

$$p_{12} = m_{11}n_{12} + m_{12}n_{22}$$

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## Matrix-Matrix Multiplication

- row by column

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$$p_{22} = m_{21}n_{12} + m_{22}n_{22}$$

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## Matrix-Matrix Multiplication

- row by column

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$$p_{11} = m_{11}n_{11} + m_{12}n_{21}$$

$$p_{21} = m_{21}n_{11} + m_{22}n_{21}$$

$$p_{12} = m_{11}n_{12} + m_{12}n_{22}$$

$$p_{22} = m_{21}n_{12} + m_{22}n_{22}$$

- noncommutative:  $AB \neq BA$

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## Matrix-Vector Multiplication

- points as column vectors: postmultiply

$$\begin{bmatrix} x' \\ y' \\ z' \\ h' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ h \end{bmatrix} \quad p' = Mp$$

- points as row vectors: premultiply

$$\begin{bmatrix} x' & y' & z' & h' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}^T \begin{bmatrix} x \\ y \\ z \\ h \end{bmatrix} \quad p'^T = p^T M^T$$

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## Matrices

- transpose
 
$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}^T = \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix}$$
- identity
 
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
- inverse  $AA^{-1} = I$ 
  - not all matrices are invertible

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## Matrices and Linear Systems

- linear system of  $n$  equations,  $n$  unknowns

$$3x + 7y + 2z = 4$$

$$2x - 4y - 3z = -1$$

$$5x + 2y + z = 1$$

- matrix form  $\mathbf{Ax}=\mathbf{b}$

$$\begin{bmatrix} 3 & 7 & 2 \\ 2 & -4 & -3 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$