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Representing Orientation

Representing Translations and Positions

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- to translate by 30 units in x:
  - *add together thirty 1 unit translations*
- arithmetic interpolation
  (divide the total translation by n)

Representing Rotations and Orientations

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- to rotate by 30 degrees:
  - *R' = R^30*
    - where R is a 3x3 or 4x4 matrix that rotates by one degree
- geometric interpolation (take the nth root of the desired final rotation matrix)

Representing Rotations and Orientations

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- how many degrees of freedom in 3D ?

- desired features of any representation
  - *unique*
  - *continuous*
  - *compact*
  - *efficient to work with*

Rotation in a 2D world

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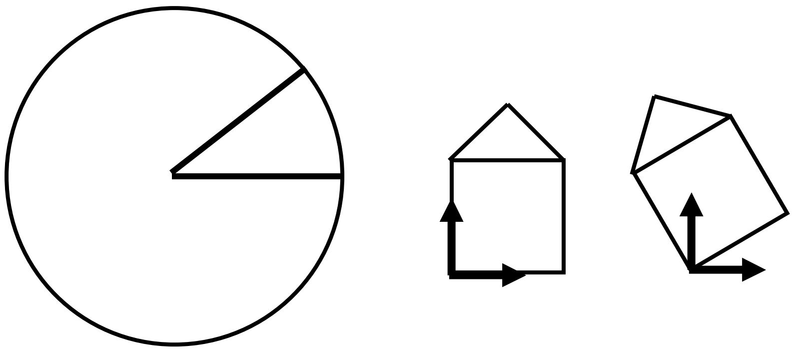

Rotation in a 3D world

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- SO(3) group in Lie algebra

- four common alternative numerical representations:
  - *3x3 rotation matrix*
  - *Euler angles (fixed angles)*
  - *exponential map*
  - *unit quaternions*

3x3 Rotation Matrix

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- 9 elements
- 3 orthogonality constraints
- renormalization algorithms
- extracting pure rotational component (polar decomp)

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$R = \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix}$$

$$R^{-1} = R^T$$

$$a \bullet b = 0 \qquad |a| = 1$$

$$b \bullet c = 0 \qquad |b| = 1$$

$$a \bullet c = 0 \qquad |c| = 1$$

... and determinant = 1

Euler Angles

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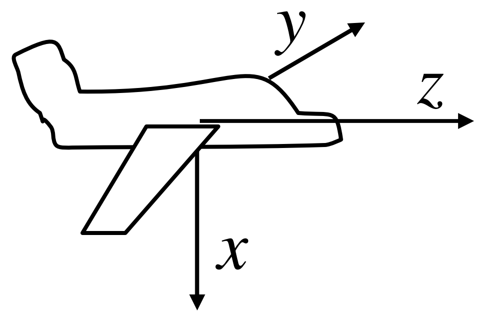
- choose 3 successive rotations about different axes
  - *e.g., RPY:* z, y, x



roll **pitch** **yaw**

$$R_{RPY} = Rot(z,\alpha)\, Rot(y,\beta)\, Rot(x,\gamma)$$

- common alternative: z, x, z
- problem: “gimbal lock”
- problem: non-uniqueness RPY(0,90,0) = RPY(90,90,90)

Euler's Rotation Theorem

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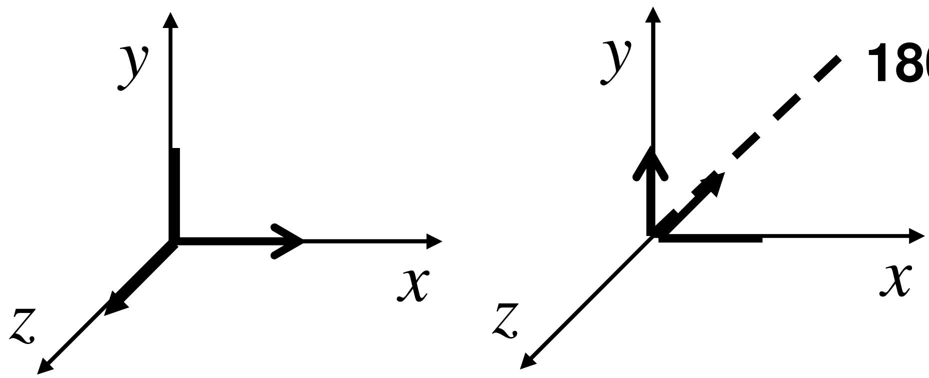
- can always go from one orientation to another with one rotation about a single axis



$$Rot(\vec{k},\theta) = \begin{bmatrix} k_x^2 v + c & k_x k_y v - k_z s & k_x k_z v + k_y s \\ k_x k_y v + k_z s & k_y^2 v + c & k_y k_z v - k_x s \\ k_x k_z v - k_y s & k_y k_z v + k_x s & k_z^2 v + c \end{bmatrix}$$

where

$$c = \cos\theta$$
$$v = 1 - \cos\theta$$
$$s = \sin\theta$$

Exponential Map

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- idea: encode amount of rotation into magnitude of $\vec{k}$

$$\left|\vec{k}\right| = \theta \qquad Rot(\vec{k}, \left|\vec{k}\right|) \qquad \Re^3 \longrightarrow SO(3)$$

- axis definition undefined for no rotation
  - *therefor define the zero vector to be the identity rotation*
- singularities for $\left|\vec{k}\right| = 2\pi n$

Unit quaternions

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$$q = w + xi + yj + zk$$

$$\begin{bmatrix} x & y & z & w \end{bmatrix} = (s, \vec{v})$$

where $$q = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{k})$$

- rotation of a vector, i.e., a point in a coord frame:

$$\vec{v}' = Rot(\vec{k}, \theta)\vec{v} = q \cdot \tilde{v} \cdot \overline{q}$$

$$\tilde{v} = (0, \vec{v}) \qquad \overline{q} = (s, -\vec{v})$$

- two successive rotations

$$q_2(q_1 \cdot \tilde{v} \cdot \overline{q}_1)\overline{q}_2$$

Quaternion Math

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$$i^2 = -1 \qquad i \cdot j = -j \cdot i = k$$
$$j^2 = -1 \qquad j \cdot k = -k \cdot j = i$$
$$k^2 = -1 \qquad k \cdot i = -i \cdot k = j$$

RH rule

$$q^{-1} = \frac{1}{\|q\|^2}[s, -v]$$

$$qq^{-1} = [1, (0,0,0)]$$

- unit quaternions

$$w^2 + x^2 + y^2 + z^2 = 1$$

- addition $(s_1, v_1) + (s_2, v_2) = (s_1 + s_2, v_1 + v_2)$
- multiplication

$$(s_1, v_1) \cdot (s_2, v_2) = (s_1 \cdot s_2 - v_1 \bullet v_2, s_1 \cdot v_1 + s_2 \cdot v_2 + v_1 \times v_2)$$

Orientation Interpolation

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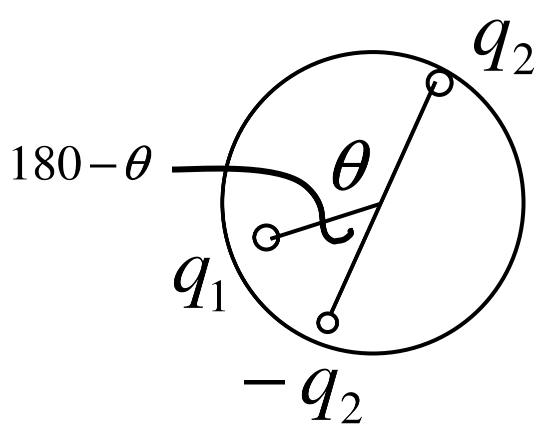
- linear interpolation of quaternions
- note: q and –q represent the same orientation

$$q_1 \longrightarrow q_2 \quad \textbf{or} \quad q_1 \longrightarrow -q_2 \quad \textbf{?}$$

choose shorter path, use dot product to compute



$$\cos\theta = q_1 \cdot q_2 = s_1 \cdot s_2 + v_1 \bullet v_2$$

Orientation Interpolation

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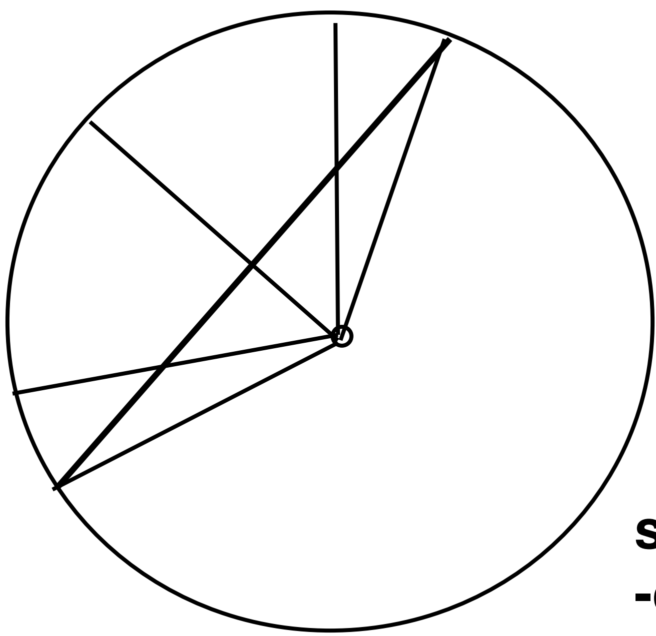
SLERP instead of LERP

$$slerp(q_1, q_2, u) = \frac{\sin((1-u)\theta)}{\sin\theta} q_1 + \frac{\sin(u\theta)}{\sin\theta} q_2$$



smooth interpolation of multiple orientations:
-construct smooth curve on the 4D sphere