Numerical Approximation and Discretization
CPSC 303, Term 2, Winter 2005–2006

Linear Algebra Review

Notation:

- Matrix (denoted by capital letters) $A \in \mathbb{R}^{m \times n}$ has m rows and n columns. Entry a_{ij} appears in row i and column j of A. When talking about matrices, remember: "rows then columns."
- Vector $v \in \mathbb{R}^n$ has n rows (a column vector). Entry v_i appears in row i.
- For $A \in \mathbb{R}^{m \times n}$ and $B \in R^{n \times r}$, $C = AB$ is $C \in R^{m \times r}$. In particular, for $v \in R^n$, $Av = u$ for $u \in \mathbb{R}^m$.
- A "lower triangular" matrix L has all zeros above the diagonal: $l_{ij} = 0$, $i < j$. An "upper triangular" matrix U has all zeros below the diagonal: $u_{ij} = 0$, $i > j$. A "diagonal" matrix D is upper and lower triangular: $D_{ij} = 0$, $i \neq j$. A "tridiagonal" matrix T has zeros except for the diagonal and the immediate sub- and super-diagonal: $t_{ij} = 0$, $i < j - 1$ or $i > j + 1$.
- The identity matrix I is diagonal with ones along the diagonal and zeros everywhere else. We will sometimes use 1 and 0 to mean appropriately sized vectors of all 1 or all 0 respectively.
- The determinant of a matrix is $\det(A)$ (often written $\det A$ without the parentheses). It is generally expensive to compute, but for triangular $A \in \mathbb{R}^{n \times n}$ (including diagonal A), $\det(A) = \Pi_{i=1}^{n} a_{ii}$ (the product of the diagonal elements of A).

Linear Systems: For $Ax = b$ where $A \in \mathbb{R}^{n \times n}$ and x, $b \in \mathbb{R}^n$. Given A and b, we will want to solve for x.

- The following are equivalent:
  1. A is singular.
  2. $\det A = 0$.
  3. $Ax = b$ may have zero or infinitely many solutions.
  4. There exists $x \neq 0$ such that $Ax = 0$.
- The following are equivalent:
  1. A is nonsingular.
  2. $\det A \neq 0$.
  3. $Ax = b$ has a unique solution.
  4. A^{-1} (the inverse of A) exists and $x = A^{-1}b$.

Conditioning: The condition number $\text{cond}(A) = \|A\|\|A^{-1}\|$. For singular A, we define $\text{cond}(A) = \infty$. Large finite condition numbers imply that $Ax = b$ is an ill-conditioned problem to solve for x, as shown below.

Let $Ax = b$, A nonsingular and define $\hat{x}$, $\hat{b}$, Δx and Δb by

$$\begin{aligned} A\hat{x} &= \hat{b}, \\ A(x + \Delta x) &= (b + \Delta b). \end{aligned}$$

Then $A\Delta x = \Delta b$ implies

$$\begin{aligned} \Delta x &= A^{-1}\Delta b, \\ \|\Delta x\| &= \|A^{-1}\Delta b\|, \\ &\leq \|A^{-1}\|\|\Delta b\|. \end{aligned}$$

Also,

$$\begin{aligned} b &= Ax, \\ \|b\| &= \|Ax\|, \\ &\leq \|A\|\|x\| \implies \|x\| \geq \|b\|/\|A\|. \end{aligned}$$

So relative error in solution $\|\Delta x\|/\|x\|$ is bounded by

$$\frac{\|\Delta x\|}{\|x\|} \leq \|A^{-1}\|\|\Delta b\|\frac{\|A\|}{\|b\|} = \text{cond}(A)\frac{\|\Delta b\|}{\|b\|}.$$

Computational Cost: The cost of solving $Ax = b$ depends on the pattern of known zeros in $A \in \mathbb{R}^{n\times n}$.

- Diagonal or tridiagonal A: $\mathcal{O}(n)$. Also applies if there are any constant number of nonzeros per row, but the algorithms are a little more complex.
- Lower or upper triangular A: $\mathcal{O}(n^2)$. This algorithm is generally called "backward" or "forward" substitution.
- Dense A (no particular pattern of nonzeros): $\mathcal{O}(n^3)$.

More details can be found in Heath chapter 2, particularly section 2.4–2.6.