Numerical Approximation and Discretization

Linear Algebra Review

Notation:

- Matrix (denoted by capital letters) $A \in \mathbb{R}^{m \times n}$ has $m$ rows and $n$ columns. Entry $a_{ij}$ appears in row $i$ and column $j$ of $A$. When talking about matrices, remember: “rows then columns.”

- Vector $v \in \mathbb{R}^n$ has $n$ rows (a column vector). Entry $v_i$ appears in row $i$.

- For $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times r}$, $C = AB$ is $C \in \mathbb{R}^{m \times r}$. In particular, for $v \in \mathbb{R}^n$, $Av = u$ for $u \in \mathbb{R}^m$.

- A “lower triangular” matrix $L$ has all zeros above the diagonal: $l_{ij} = 0$, $i < j$. An “upper triangular” matrix $U$ has all zeros below the diagonal: $u_{ij} = 0$, $i > j$. A “diagonal” matrix $D$ is upper and lower triangular: $D_{ij} = 0$, $i \neq j$. A “tridiagonal” matrix $T$ has zeros except for the diagonal and the immediate sub- and super-diagonal: $t_{ij} = 0$, $i < j - 1$ or $i > j + 1$.

- The identity matrix $I$ is diagonal with ones along the diagonal and zeros everywhere else. We will sometimes use 1 and 0 to mean appropriately sized vectors of all 1 or all 0 respectively.

- The determinant of a matrix is $\det(A)$ (often written $\det A$ without the parentheses). It is generally expensive to compute, but for triangular $A \in \mathbb{R}^{n \times n}$ (including diagonal $A$), $\det(A) = \Pi_{i=1}^n a_{ii}$ (the product of the diagonal elements of $A$).

**Linear Systems:** For $Ax = b$ where $A \in \mathbb{R}^{n \times n}$ and $x, b \in \mathbb{R}^n$. Given $A$ and $b$, we will want to solve for $x$.

- The following are equivalent:
  1. $A$ is singular.
  2. $\det A = 0$.
  3. $Ax = b$ may have zero or infinitely many solutions.
  4. There exists $x \neq 0$ such that $Ax = 0$.

- The following are equivalent:
  1. $A$ is nonsingular.
  2. $\det A \neq 0$.
  3. $Ax = b$ has a unique solution.
  4. $A^{-1}$ (the inverse of $A$) exists and $x = A^{-1}b$. 

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**Conditioning:** The condition number \( \text{cond}(A) = \|A\|\|A^{-1}\| \). For singular \( A \), we define \( \text{cond}(A) = \infty \). Large finite condition numbers imply that \( Ax = b \) is an ill-conditioned problem to solve for \( x \), as shown below.

Let \( Ax = b \), \( A \) nonsingular and define \( \hat{x}, \hat{b}, \Delta x \) and \( \Delta b \) by

\[
A\hat{x} = \hat{b},
\]
\[
A(x + \Delta x) = (b + \Delta b).
\]

Then \( A\Delta x = \Delta b \) implies

\[
\begin{align*}
\Delta x &= A^{-1}\Delta b, \\
\|\Delta x\| &= \|A^{-1}\Delta b\|, \\
&\leq \|A^{-1}\|\|\Delta b\|.
\end{align*}
\]

Also,

\[
\begin{align*}
b &= Ax, \\
\|b\| &= \|Ax\|, \\
&\leq \|A\|\|x\| \implies \|x\| \geq \|b\|/\|A\|.
\end{align*}
\]

So relative error in solution \( \|\Delta x\|/\|x\| \) is bounded by

\[
\frac{\|\Delta x\|}{\|x\|} \leq \|A^{-1}\|\|\Delta b\|\frac{\|A\|}{\|b\|} = \text{cond}(A) \frac{\|\Delta b\|}{\|b\|}.
\]

**Computational Cost:** The cost of solving \( Ax = b \) depends on the pattern of known zeros in \( A \in \mathbb{R}^{n \times n} \).

- Diagonal or tridiagonal \( A \): \( O(n) \). Also applies if there are any constant number of nonzeros per row, but the algorithms are a little more complex.
- Lower or upper triangular \( A \): \( O(n^2) \). This algorithm is generally called “backward” or “forward” substitution.
- Dense \( A \) (no particular pattern of nonzeros): \( O(n^3) \).

More details can be found in Heath chapter 2, particularly section 2.4–2.6.