# CPSC 490 – Problem Solving in Computer Science

## Lecture 24: Konig's Theorem

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2017/03/22

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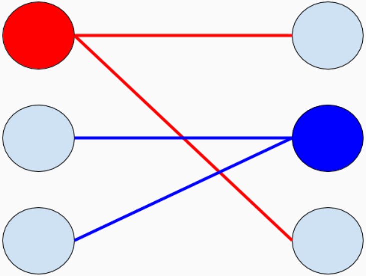
# Last Time

- Maximum Bipartite Matching

# Vertex Cover

Given a graph $G = (V, E)$, a vertex cover $C \subset V$ is such that all edges $e \in E$ have $\geq 1$ endpoint in $C$.

We are generally interested in the minimum vertex cover.



Size of the minimum vertex cover is 2.

# Vertex Cover

Min-VC is an NP-Hard problem in general.

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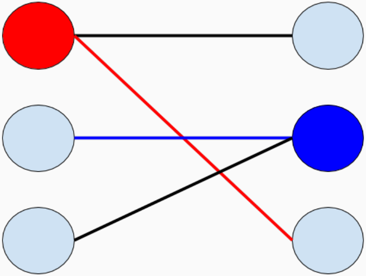
In the case of bipartite graphs, we have hope, thanks to our friend Konig!

# Vertex Cover in Bipartite Graphs

Every edge in the matching needs to have $\geq 1$ endpoint in the cover.
$\Rightarrow$ Size of any vertex cover $\geq$ the size of any matching.

# Vertex Cover in Bipartite Graphs

The minimum vertex cover is 2. The size of the maximum matching in this graph is also 2!



Size of the maximum matching is 2.

# Konig’s Theorem

**Theorem (Konig’s Theorem)**

*In a bipartite graph, the size of the maximum matching is equal to the size of the minimum vertex cover.*

# Proof 1

We can prove this using Linear Programming Duality!

# Proof 1

First, let's look at the primal program for bipartite matching:

$$\begin{aligned} \text{maximize:} \quad & \sum_{uv \in E} x_{uv} & \\ \text{subject to:} \quad & \sum_{v \in V} x_{uv} \leq 1 & \forall u \in U \\ & \sum_{u \in U} x_{uv} \leq 1 & \forall v \in V \\ & x_{uv} \geq 0 & \forall uv \in E \end{aligned}$$

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\end{aligned}
$$

- Each primal variable $x_{uv}$ corresponds to an edge $u - v$ in graph.
- $uv \in M \leftrightarrow x_{uv} = 1$, otherwise $x_{uv} = 0$.
- Objective: maximize the size of the matching.

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Now, let's add a dual variable $y_u$ for each constraint and look at the dual linear program.

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- Each dual variable $y_u$ corresponds to a vertex $u \in G$.
- $y_u = 1$ or $0$. Think: $y_u = 1 \leftrightarrow u$ is in the vertex cover.
- Objective: minimize sum of $y_i$ s.t. each edge has $\geq 1$ endpoint having value 1.

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- By Linear Programming Duality, the value of the two optimal solutions is equal.
- $\max \sum_{uv \in E} x_{uv} = \min \sum_{u \in U} y_u + \sum_{v \in V} y_v$.

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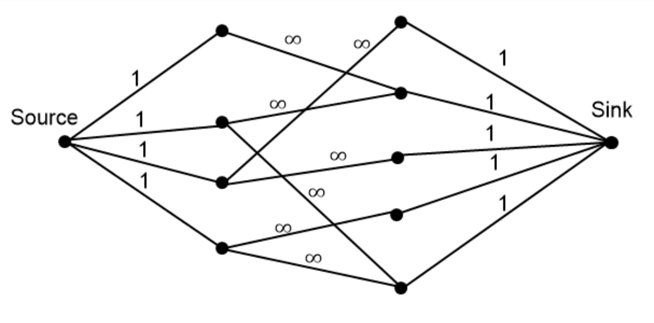
- $\max \sum_{uv \in E} x_{uv} = \min \sum_{u \in U} y_u + \sum_{v \in V} y_v$.

In other words, the size of the maximum matching is equal to the size of the minimum vertex cover.

# Proof 2

Cut edge is either $s \to u$ or $v \to t$
$\Rightarrow$ Minimum cut = minimum vertex cover!



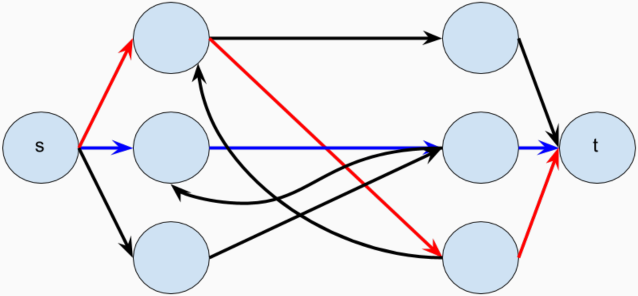

Suppose not, suppose $u \to v$ is not covered, then there is path $s \to u \to v \to t$ so we did not have a valid cut.

# Finding the Cover

```
Find the maximum matching in G
DFS(source) in the residual graph
Let L = reachable vertices
Output C = (U \ L) ∪ (V ∩ L)
```

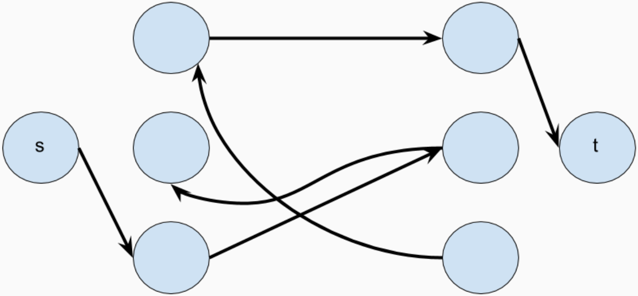
Proof: same as finding min cut in flow

# Finding the Cover



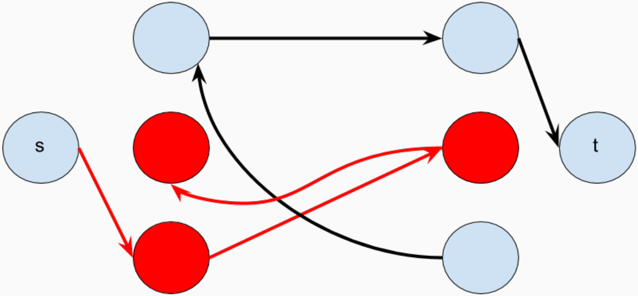

Residual graph after finding the max flow.
Black edges have residual capacity = 1.
Colored edges have flow = 1, so residual
capacity = 0.

# Finding the Cover



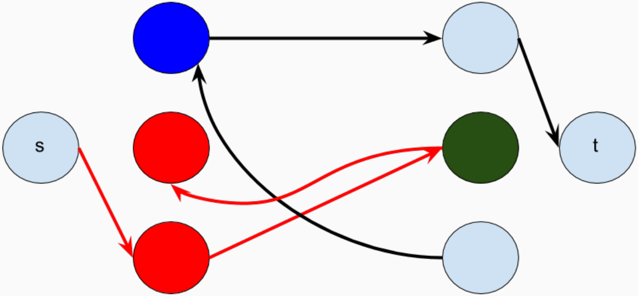

Residual graph after finding the max flow.
Only showing the edges with residual
capacity = 1.

# Finding the Cover



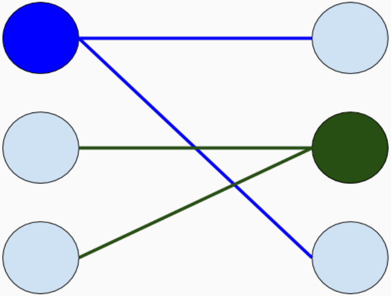

L = red vertices, ie. reachable from source in the residual graph.

# Finding the Cover




A - L = blue vertices.
B & L = green vertices.
L = red vertices, ie. reachable from source in the residual graph.

# Finding the Cover



All edges have at least 1 endpoint that's blue or green.

# Proof of Correctness

Proof that $C = (U \setminus L) \cup (V \cap L)$ is a vertex cover:

**Proof.**

- Suppose $C$ is not a vertex cover.
- Then $\exists e = (u, v) \in E$ s.t. $u \in U \cap L$ and $v \in V \setminus L$.

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- $e \notin M$. If $e \in M$, then $v \in L$ otherwise $u$ would not be in $L$.
- Thus, $e \in E \setminus M$.

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- $e \notin M$. If $e \in M$, then $v \in L$ otherwise $u$ would not be in $L$.
- Thus, $e \in E \setminus M$.
- But then, $v$ is reachable from $u$ - go from source to $u$, and take the edge to $v$ since it has residual capacity = 1.
- $\Rightarrow v \in L$ but this contradicts that $v \notin L$.

$\square$

# Proof of Correctness

Proof that $|C| \leq |M|$:

**Proof.**

- No vertex in $U \setminus L$ is unmatched.
- No vertex in $V \cap L$ is unmatched.
- There is no $e = (u, v) \in M$ s.t. $u \in U \setminus L$ and $v \in V \cap L$ since otherwise, $u \in L$.

$\Rightarrow$ every vertex in $C$ is matched and corresponding edges of matching are distinct. So $|C| \leq |M|$. $\square$

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$\Rightarrow$ every vertex in $C$ is matched and corresponding edges of matching are distinct. So $|C| \leq |M|$. $\square$

We already know that $|C| \geq |M|$ for any cover and any matching. Thus, $|C| = |M|$.

# Proof of Correctness

Thus, $C = (U \setminus L) \cup (V \cap L)$ is a minimum vertex cover.

# Maximum Independent Set

A set of vertices is an independent set if there are no edges between any pair of them.

A maximum independent set is the largest such set

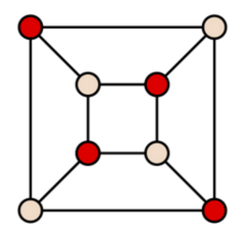

**Figure 1:** Red vertices form a maximum independent set. *Source: Wikipedia*

NP complete in general, but easy for bipartite graphs! Thanks Konig!

# Minimum Vertex Cover $\Leftrightarrow$ Maximum Independent Set

This is because complement of any vertex cover is independent set and vice versa

Proof

- ($\Rightarrow$) if $u$ and $v$ are both not in vertex cover then there cannot be an edge between them
- ($\Leftarrow$) for any edge $u \rightarrow v$, at least one of $u$ and $v$ not in independent set

# Minimum Edge Cover

An edge cover is a set of edges such that every vertex is an endpoint of $\geq 1$ edge in the edge cover.

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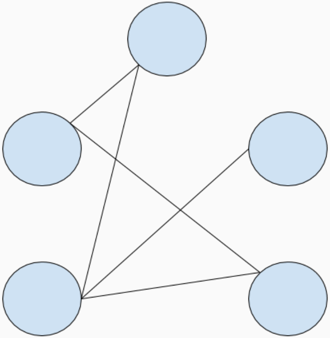
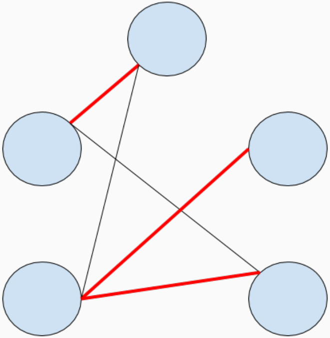
A minimum edge cover an edge cover of the smallest possible size.

# Minimum Edge Cover



# Minimum Edge Cover



# Minimum Edge Cover

Note: There is no edge cover in a graph with isolated vertices.

# Minimum Edge Cover $\Leftrightarrow$ Maximum Matching

In particular, let $E =$ size min edge cover, $M$ = size of matching

$$E = V - M$$

Proof

- Take any matching, greedily add one edge from every unmatched vertex, then we must get edge cover, so $E \leq M + (V - 2M) = V - M$
- Any minimum edge cover must not have path of length $\geq 3$ (otherwise can delete some edges) $\Rightarrow$ must be graph of stars
  - Each star of $q$ vertices has $q - 1$ edges $\Rightarrow$ # vertices = # stars + # edges, so $V = k + E$
  - Construct matching by picking 1 edge from each star, then we get $k \leq M$, so $E = V - k \geq V - M$

# Minimum Edge Cover

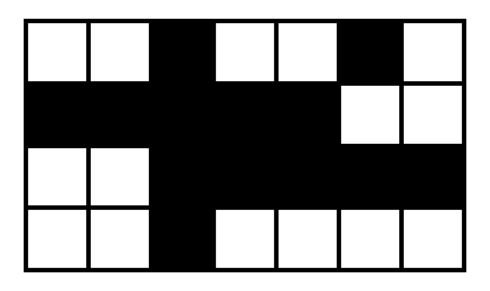
```
Find the maximum matching in G
Add rest of the edges greedily
```

*G* is any graph, not necessarily bipartite.

# Problem 1

You have unlimited tiles of size $1 \times k$ and $k \times 1$ for all $k = 1, 2, 3, \ldots$.

Use a minimum number of them to cover all the black tiles while not touching any white tiles. Tiles may overlap with each other.



# Problem 1 - Solution

Observation 1: want to expand every tile as much as possible – no point wasting two tiles adjacent in the same direction.

$\Rightarrow$ Every black cell covered by at most 2 tiles – one horizontal and one vertical

$\Rightarrow$ Form bipartite graph where $i \to j$ if intersection of horizontal tile $i$ and vertical tile $j$ is a black cell, find max vertex cover

# Problem 2

TV show invite speakers for Trump and speakers for Hillary to debate.

After the first round of debate, viewers cast two votes:

- One for their favorite speaker
- One for their least favorite speaker.

Of course, we all know people support either Trump or Hillary, so

- If a viewer is a Trump supporter, he/she will vote for a Trump speaker and against a Hillary speaker.

Viewers are happy if in the next round, their favorite speaker is kept in the show and their least favorite speaker is removed.

What's the maximum number of happy viewers we can keep?

# Problem 2 - Solution

Form a graph where there is edge $i \to j$ if viewers $i$ and $j$ have conflicting votes.

Observe that this graph is bipartite!

- Hillary supporters never vote against each other
- Trump supporters never vote against each other

Need to find maximum number of viewers that don't conflict with each other $\Rightarrow$ maximum independent set = $n-$min vertex cover

# Next Class

More Flow Practice, Flow With Demands