CPSC 490 – Problem Solving in Computer Science

Lecture 12: SQRT Decomposition

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Based on slides by Paul Liu (2014), Kuba Karpierz and Bruno Vacherot (2015)

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Range Query

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- Median
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- Median
- Mode
- What if we allow updates?

Range Minimum Query with Updates

Given an array A of n integers, perform these queries efficiently:

- `update(i, x)`: set `A[i] = x`
- `min(i, j)`: return `min(A[i], A[i+1], ... , A[j])`

Potential Solutions

We could try the following:

- Option 1: Naively perform the 2 queries.
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Can we do better? Can we do a different type of precomputation that allows us to `update` and `min` fast?

Better Solution

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- `update(i, x)`: Update the minimum in the relevant block in $O(n/k)$ time (since each block has n/k elements).
- `min(i, j)`: Lookup the relevant minimums from $O(k)$ blocks.

But how do we choose k?

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Divide the array into $\sqrt{n}$ blocks!

- If $\sqrt{n}$ is not an integer, just round down or up, doesn't matter.
- The last block may have size $< \sqrt{n}$, that's fine.

Square Root Decomposition

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- `update(i, x)`: Go to the block $\lfloor i/\sqrt{n} \rfloor$. Set `A[i] = x` and update minimum in that block. $O(\sqrt{n})$.
- `min(i, j)`: Determine the start and end blocks.
    - If start/end lie in the middle of a block, brute force those sections.
    - For all the blocks in the middle, lookup the precomputed answer.
    - We loop over at most $2\sqrt{n}$ elements and $\sqrt{n}$ blocks. $O(\sqrt{n})$.

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What do we store in each block? The sum for that block.

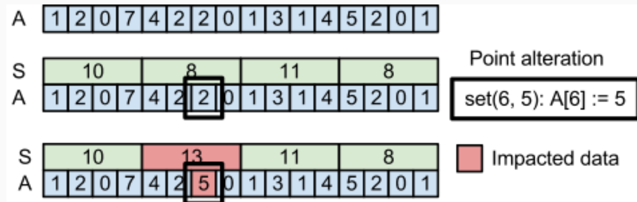
SQRT Decomposition - Example

`sum(2, 14)`:

Source: `http://www.infoarena.ro/blog/square-root-trick`

SQRT Decomposition - Example

`update(6, 5)`:



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Queries Without Updates

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What about range query problems without updates?

If there are no updates, we can process queries in a different order, and return the results all at once!

In other words, we do queries offline!

Square Root Decomposition of Queries

Observation 1: if all queries have the same left end point, then we can sort by right end point and then do linear time sweep!

- Min queries: if we know `min(a, b)` and we want `min(a, c)`, just min with elements from index $b + 1$ to c.

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Observation 2: if left end points differ by a little, then moving the left end back and forth is not too expensive, so our strategy still works.

- Min queries (again): to get from `min(a, b)` to `min(c, d)`, keep a set elements from a to b, then add elements from $b+1$ to d, and add/delete elements between a and c.

Square Root Decomposition of Queries

Idea: Sort the queries by the $\sqrt{n}$ block their left endpoint is in, and then by their right endpoint.

Query $(a, b) < (c, d)$ iff:

- $\lfloor a/\sqrt{n} \rfloor < \lfloor c/\sqrt{n} \rfloor$ or
- $\lfloor a/\sqrt{n} \rfloor = \lfloor c/\sqrt{n} \rfloor$ and $b < d$.

Square Root Decomposition of Queries

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For min queries, an balanced BST storing the set of elements in the current interval will do the job in $O(\log n)$ per element change.

Square Root Decomposition of Queries

Number of element changes

- Left end point moves $\leq \sqrt{n}$ times per query
- Right end point moves $\leq n$ times per block

$\Rightarrow O(q\sqrt{n} + n\sqrt{n})$ element changes in total

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Time complexity for min query

- Sorting queries take $O(q \log q)$
- For min query, each element change costs $O(\log n)$

$\Rightarrow O(q \log q + (n + q)\sqrt{n} \log n)$

Problem 1

Given array A of $n \leq 10,000$ ints, and an int $0 \leq k \leq 1,000,000$.

Answer $q \leq 10000$ queries of the following form:

For query (a, b), return the number of pairs of indices i, j such that

- $a \leq i < j \leq b$
- $A[i] \oplus A[i+1] \oplus \cdots \oplus A[j] = k$.

where $\oplus$ is the bit-wise XOR operator.

Hint: bit-wise XOR of integers = addition in vector space of {0, 1} and addition = subtraction when you only have two numbers 0 and 1

Problem 1 - Solution

No updates $\Rightarrow$ let's use square root decomposition for offline query

From answer to query (a, b), how do we get answer for $(a, b + 1)$?

Problem 1 - Solution

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Also, note that $P[i] \oplus P[j] = k \Leftrightarrow P[j] = k \oplus P[i]$.

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- $\Rightarrow$ add $cnt[k \oplus P[b+1]]$ to result and increment $cnt[P[b+1]]$

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Remove an element $A[a]$ from left end of interval

- Look at how many $k \oplus P[a-1]$ we have in $P[a], \ldots, P[b]$
- $\Rightarrow$ subtract $cnt[k \oplus P[a-1]]$ from result and decrement $cnt[P[a-1]]$, beware when $P[a-1] = k \oplus P[a-1]$

Addition on the left end is handled similarly.

Problem 2

You are given an array A with $n \leq 10,000$ integers.

Answer $q \leq 10,000$ queries of the form *mode(l,r)*.

No updates.

Problem 2 - Solution

No updates $\Rightarrow$ let's use square root decomposition for offline query

But how to get mode of $[a_{t+1}, b_{t+1}]$ given mode of $[a_t, b_t]$?

Problem 2 - Solution

Keep track of two things:

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If we have the above information for an interval $[a, b]$, then to go to e.g. $[a, b+1]$ we just need to insert 1 element:

- Erase (new element, count) from the set
- Increment count of the new element
- Insert (new element, new count) back to the set

Deletion is pretty similar

Next Class

Segment Tree