Rigid Body Dynamics

skew-symmetric matrix

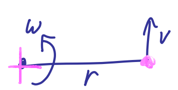
Definition: $\vec{a} \times \vec{b} = \tilde{a} b$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$a \times b = -b \times a$$
$$= -\tilde{b} a$$

Dynamics of Rotation

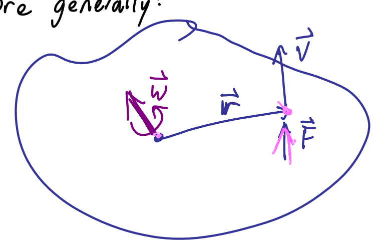
Intuitively, with scalars:



$v = w \cdot r$

$\tau = r \cdot F$

More generally:



$\vec{V} = \vec{\omega} \times \vec{r}$ velocity

$\vec{\tau} = \vec{r} \times \vec{F}$ torque

$\vec{P} = m \cdot \vec{V}$ linear momentum

$\vec{L} = r \times \vec{p}$ angular momentum

linear momentum

Newton's Law

$$\frac{d(ab)}{dt} = \frac{da}{dt}b + a\frac{db}{dt}$$

Conservation of linear momentum

$$\sum F = \frac{d\vec{p}}{dt} \quad \text{where } \vec{p} = m\cdot\vec{v}$$

$$= \frac{d}{dt}(m\cdot\vec{v}) = \cancel{\dot{m}\vec{v}}^{\,0} + m\dot{\vec{v}}$$

$$= m\dot{v}$$

$$= ma$$

Conservation of angular momentum

$$\sum \tau = \frac{dL}{dt} \quad \text{where } L = \text{angular momentum}$$

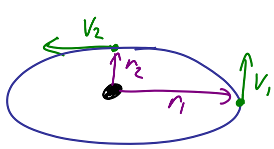
$$L = \underline{I\omega}$$

$$= \frac{d}{dt}(I\omega) = \dot{I}\omega + I\dot{\omega}$$

$$= \underline{\omega \times I\omega} + I\dot{\omega}$$

Examples : Conservation of angular momentum

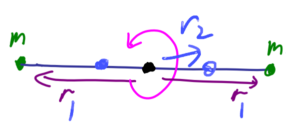
①



$$\underline{L} = \underline{m\cdot v\cdot r}$$

$$\uparrow \; \downarrow$$

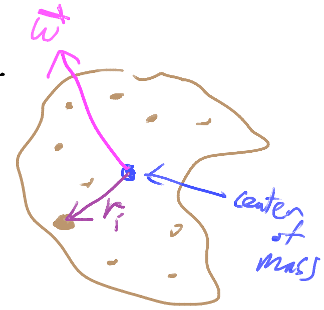
②



$$m\cdot v_1\cdot r_1 = m\, v_2\cdot r_2$$

$$\uparrow \; \downarrow$$

Angular momentum for a set of particles



$$\vec{L} = \sum_i r_i \times \vec{P}_i$$

$$= \sum_i r_i \times (m_i v_i) \qquad v_i = \omega \times \vec{r}_i$$

$$= \sum_i r_i \times m_i (\omega \times r_i)$$

$$= \sum_i \tilde{r}_i \, m_i (\omega \times r_i)$$

$$= \sum_i \tilde{r}_i \, m_i (-r_i \times \omega)$$

$$= \sum_i \tilde{r}_i \, m_i (-1) \tilde{r}_i \, \omega$$

$$= \underbrace{\sum_i -m_i \tilde{r}_i \tilde{r}_i}_{I} \, \omega$$

$$\vec{L} = I \vec{\omega}$$

I inertia "tensor" 3x3 matrix

analogous to: $\vec{P} = m\vec{v}$

$$\vec{L} = -\sum m_i \tilde{r}_i \tilde{r}_i \vec{\omega}$$

repeat

$$= -\sum m_i \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \vec{\omega}$$

$$= \underbrace{\sum_i m_i \begin{bmatrix} z_i^2 + y_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & z_i^2 + x_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{bmatrix}}_{I} \vec{\omega}$$

for axis symmetric objects

$$I = \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix}$$

Newton-Euler Equations of Motion

(only for inertial frames)

$$\Sigma F = \frac{dp}{dt} = \frac{d(mv)}{dt} = \cancel{\dot{m} v} + m\dot{v}$$

$$= m \cdot \vec{a}$$

$$\Sigma \tau = \frac{dL}{dt} = \frac{d(I\omega)}{dt} = \dot{I}\omega + I\dot{\omega}$$

$$= \vec{\omega} \times (I\omega) + I\dot{\omega}$$

Newton-Euler equation in matrix form

"Forward Dynamics"

solve for these

$$\begin{bmatrix} \begin{matrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{matrix} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \\ \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \Sigma F \\ (\Sigma \tau) - \omega \times (I\omega) \end{bmatrix}$$

→ valid in inertial frame

Computing I

- eqns motions require knowing I_W : i.e., I in the inertial or world frame.

$r_W = R\, r_L$ transforming vector

① $\vec{L}_W = R\, \vec{L}_L$

② $\vec{\omega}_W = R\, \vec{\omega}_L$

} R transforms vector quantities from local to world.

③ $L_W = I_W\, \omega_W$

④ $L_L = I_L\, \omega_L$

} $L = I\omega$ in both W and L frames.

(a) $L_W = R\, I_L\, \omega_L$

(b) $\omega_L = R^{-1}\, \vec{\omega}_W$

$L_W = \boxed{R\, I_L\, R^{-1}}\, \omega_W$

$I_W = R\, I_L\, R^T$

$R^{-1} \equiv R^T$

Simulation Loop

for each time step

setup – compute all forces & torques

$I_W = R\, I_L\, R^T$

solve eqns of motion – $\sum F = m \cdot a$

$\sum \tau = \omega \times I_W + I\dot{\omega}$

integration – $\vec{X} = \vec{X} + \vec{V}\Delta t$

$\vec{V} = \vec{V} + \vec{a}\,\Delta t$

$\vec{R} = \vec{R} + \dot{R}\,\Delta t$ where $\dot{R} = \tilde{\omega} R$

skew symmetric matrix

$\vec{\omega} = \vec{\omega} + \dot{\vec{\omega}}\,\Delta t$

end for

state:

- $\vec{X}$: position
- $\vec{V}$: velocity
- R or q or Euler angles : orientation
- $\vec{\omega}$: angular velocity

Simulation – Alternate choice of state

state: $\vec{x}$: position
$\vec{v}$: velocity
R : orientation
$\vec{L}$: angular momentum

for each time step

- compute forces, torques
- equations of motion

$$\Sigma F = m \cdot \vec{a}$$
$$\Sigma \tau = \dot{L}$$

- integration

$$\vec{x} = \vec{x} + \vec{v}\Delta t$$
$$\vec{v} = \vec{v} + \vec{a}\Delta t$$
$$R = R + \dot{R}\Delta t \leftarrow \dot{R} = \tilde{\omega} R$$
$$\vec{L} = \vec{L} + \dot{\vec{L}}\Delta t$$

$$\omega = I_w^{-1} L$$

Simulation – Using quaternions

(fyi only: not covered in class)

$q = q + \dot{q}\Delta t$ and then renormalize

where $\dot{q} = 0.5\, \hat{\omega} \otimes q$ and $\hat{\omega}$ is the quaternion $(0, \omega)$

and $\omega = I_w^{-1} L$