Notes

- Demetri Terzopoulos talk: Thursday, 4pm
  Dempster 310

Back to Rigid Bodies

- Motivation - particle simulation doesn’t cut it for large rigid objects
  - Especially useful for action in games and film (e.g. car dynamics, crashes, explosions)
- To recap:
  - Split our rigid body into chunks of matter, we look at each chunk as a simple particle
  - Rigid constraint: distances between particles have to stay constant
  - Thus position of a particle is a rotation + translation from “object space” into “world space”
  - We want to figure out what’s happening with velocities, forces, ...

Rigid Motion

- Recall we map from object space position $p_i$ of particle $i$ to world space position $x_i$ with $x_i = R(t)p_i + X(t)$
- Differentiate map w.r.t. time (using dot notation): $v_i = \dot{R}p_i + V$
- Invert map for $p_i$: $p_i = R^T(x_i - X)$
- Thus: $v_i = \dot{R}R^T(x_i - X) + V$
- 1st term: rotation, 2nd term: translation
  - Let’s simplify the rotation

Skew-Symmetry

- Differentiate $RR^T=\delta$ w.r.t. time:
  $$\dot{RR}^T + R\dot{R}^T = 0 \Rightarrow \dot{RR}^T = -(\dot{RR}^T)^T$$
  - Skew-symmetric! Thus can write as:
    $$\dot{RR}^T = \begin{pmatrix} 0 & -\omega_2 & \omega_3 \\ \omega_2 & 0 & -\omega_0 \\ -\omega_3 & \omega_0 & 0 \end{pmatrix}$$
  - Call this matrix $\omega^*$ (built from a vector $\omega$)
  $$\dot{RR}^T = \omega^* \Rightarrow \dot{R} = \omega^* R$$
The cross-product matrix

- Note that:
  \[
  \omega^* x = \begin{pmatrix}
  0 & -\omega_2 & \omega_1 \\
  \omega_2 & 0 & -\omega_0 \\
  -\omega_1 & \omega_0 & 0
  \end{pmatrix}
  \begin{pmatrix}
  x_0 \\
  x_1 \\
  x_2
  \end{pmatrix}
  = \begin{pmatrix}
  \omega_1 x_2 - \omega_2 x_1 \\
  \omega_2 x_0 - \omega_0 x_2 \\
  \omega_0 x_1 - \omega_1 x_0
  \end{pmatrix}
  = \omega \times x
  \]
- So we have:
  \[
  v_i = \omega \times (x_i - X) + V
  \]
- \( \omega \) is the angular velocity of the object

Angular velocity

- Recall:
  - \(|\omega|\) is the speed of rotation (radians per second)
  - \(\omega\) points along the axis of rotation (which in this case passes through the point X)
  - Convince yourself this makes sense with the properties of the cross-product

Force

- Take another time derivative to get acceleration:
  \[
  a_i = \dot{v}_i = \ddot{R}p_i + A
  \]
- Use F=ma, sum up net force on system:
  \[
  \sum_i F_i = \sum_i m_i a_i = \sum_i m_i (\ddot{R}p_i + A)
  = \ddot{R} \sum_i m_i p_i + A \sum_i m_i
  \]
- Let the total mass be \( M = \sum_i m_i \)
- How to simplify the other term?

Centre of Mass

- Let’s pick a new object space position:
  \[
  p_i^{\text{new}} = p_i - \frac{\sum_j m_j p_j}{M}
  \]
  - The mass-weighted average of the positions is the centre of mass
  - We translated the centre of mass (in object space) to the point 0
- Now:
  \[
  \sum_i m_i p_i = 0
  \]
**Force equation**

- So now, assuming we’ve set up object space right (centre of mass at 0), $F=MA$
- If there are no external forces, have $F=0$
  - Internal forces must balance out, opposite and equal
  - Thus $A=0$, thus $V=$constant
- If there are external forces, can integrate position of object just like a regular particle!

**What about R?**

- How does orientation change?
- Think about internal forces keeping the particles in the rigid configuration
  - Conceptual model: very stiff spring between every pair of particles, maintaining the rest length
  - So $F_i = \sum_j f_{ij}$ where $f_{ij}$ is force on $i$ due to $j$
- Of course $f_{ij} + f_{ji} = 0$
- Also: $f_{ij}$ is in the direction of $x_i-x_j$
  - Thus $(x_i - x_j) \times f_{ij} = 0$

**Net Torque**

- Play around: $(x_i - X) \times (x_j - X) \times f_{ij} = 0$
  - $(x_i - X) \times f_{ij} = (x_j - X) \times f_{ij} = -(x_j - X) \times f_{ji}$
- Sum both sides (look for net force)
  - $\sum_{i,j} (x_i - X) \times f_{ij} = -\sum_{i,j} (x_j - X) \times f_{ji}$
  - $\sum_i (x_i - X) \times F_i = -\sum_j (x_j - X) \times F_j$
  - $= 0$
- The expression we just computed $= 0$ is the net torque on the object

**Torque**

- The torque of a force applied to a point is $\tau_i = (x_i - X) \times F_i$
- The net torque due to internal forces is $0$
  - [geometry of torque: at CM, with opposite equal force elsewhere]
- Torque obviously has something to do with rotation
- How do we get formula for change in angular velocity?
Angular Momentum

- Use F=ma in definition of torque:
  \[ \tau_i = (x_i - X) \times m_i a_i \]
  \[ = \frac{d}{dt}[m_i(x_i - X) \times v_i] \]
- force=rate of change of linear momentum,
  torque=rate of change of angular momentum
- The total angular momentum of the object is
  \[ L = \sum_i m_i(x_i - X) \times v_i \]
  \[ = \sum_i m_i(x_i - X) \times (v_i - \dot{V}) \]

Getting to \( \omega \)

- Recall \( v_i - \dot{V} = \omega \times (x_i - X) \)
- Plug this into angular momentum:
  \[ L = \sum_i m_i(x_i - X) \times (\omega \times (x_i - X)) \]
  \[ = -\sum_i m_i(x_i - X) \times ((x_i - X) \times \omega) \]
  \[ = -\sum_i m_i(x_i - X)^* (x_i - X)^* \omega \]
  \[ = \left( \sum_i m_i(x_i - X)^* (x_i - X)^t \right) \omega \]
  \[ = I(t) \]

Inertia Tensor

- \( I(t) \) is the inertia tensor
- Kind of like “angular mass”
- Linear momentum is \( mv \)
- Angular momentum is \( L=I(t) \omega \)
- Or we can go the other way: \( \omega=I(t)^{-1}L \)

Equations of Motion

- \( \frac{d}{dt}V = \frac{F}{M} \)
- \( \frac{d}{dt}L = T \)
- \( \frac{d}{dt}X = V \)
- \( \omega = I(t)^{-1}L \)
- \( \frac{d}{dt}R = \omega^* R \)

In the absence of external forces \( F=0, T=0 \)
Reminder

- Before going on:
- Remember that this all boils down to particles
  - Mass, position, velocity, (linear) momentum, force are fundamental
  - Inertia tensor, orientation, angular velocity, angular momentum, torque are just abstractions
  - Don’t get too puzzled about interpretation of torque for example: it’s just a mathematical convenience