Reductions Redux

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- Reductions in Java
- A Hierarchy of hard problems

Reductions in Java

- A language is a set of strings. Let A be a language.
- A java method that takes a string as an argument and returns a boolean can be a decider or a recognizer for a language.
  - If the method returns `true` for every string in the language and returns `false` for every string that is not in the language, then that method is a decider.
  - If the method returns `true` for every string in the language and returns `false` or "loops" for every string that is not in the language, then that method is a recognizer.

Previous Results, $HALT$

From previous lectures, we know that:

- There is no TM that decides $HALT$ where

$$HALT \quad = \quad \{M\#w \mid M \text{ is a TM that halt when run with input } w\}$$

  - By the equivalence of Java programs and TMs, we conclude that there is no Java method that decides $HALT$ (or any other programming language).
  - Furthermore, we can define

$$HALT_J \quad = \quad \{J\#s \mid J \text{ is a Java program that halts when run with input } s\}$$

    For simplicity, we'll assume that the "input" to a Java program is a string. Note that we can take any collection of parameters of any types and represent them with strings.
  - $HALT_J$ cannot be decided by any TM or Java program (or any program in any other language).
  - $HALT$ and $HALT_J$ can be recognized by a TM or a Java program.

Previous Results, A_{TM} and E_{TM}

- $A_{TM} = \{M\#w \mid \text{TM } M \text{ accepts string } w\}$.
  - A_{TM} is Turing and Java recognizable but is neither Turing nor Java decidable.
  - We can define A_{Java} in the obvious manner, and it is Turing and Java recognizable, but neither Turing nor Java decidable.
- $E_{TM} = \{M \mid L(M) = \emptyset\}$.
  - E_{TM} is Turing and Java co-recognizable, but is neither Turing nor Java decidable.
  - This means that there is a TM (equivalently Java program) that rejects every string that is not in E_{TM} and for any string in E_{TM} either accepts or loops.
  - We can define E_{Java} in the obvious manner, and it is Turing and Java co-recognizable, but neither Turing nor Java decidable.

Some handy Java methods

I'll assume that we have the following Java methods available:

```
String[] getArgs(String s) {
    /* return an array, args, such that
     *    s = args[0] + '#' + args[1] + ...
     *                    '#' + args[args.length-1]
     * and none of the args strings contain the '#' character.
     */
}

boolean simulate(String J, String s) {
    /* Simulate java program J running with input s.  */
    /* If J not a valid program, return(false).  */
    /* Else if J accepts s, return true.  */
    /* Else if J rejects s, return false.  */
    /* Else (J loops on s) never return.  */
}
boolean simulate(String s) {
    String[] args = getArgs(s);
    if(args.length ≠ 2) return(false);
    return(simulate(args[0], args[1]));
}
```

Some more handy methods

```
boolean anbn(String s) {
    // return true if there is an integer, n, such that s = a^n b^n.  ..
}
```

$REGULAR$ and Java

- $REGULAR = \{M \mid L(M)$ is regular, M describes a TM$\}$.
- $REGULAR_J = \{J \mid L(J)$ is regular, J is the source code of a Java program$\}$.
- Reducing A_J to $REGULAR$ (using Java)
  - Assume we have a method

    ```
    boolean regularJ(String J) { ...}
    ```

    That decides language $REGULAR_J$.
  - We use `regular` to write a Java method that decides J.

    ```
    boolean aJ(String s) { /* return true if s = J#w and J accepts w
        return(regular(
               "boolean foo(String x) {"
            + "   if(anbn(x))"
            + "      return(true);"
            + "   else"
            + "      return(simulate(" + s + "));"
            + "}"
        )); }
    ```

$REGULAR_J$ (cont)

- From previous slide

```
boolean aJ(String s) { /* return true if s = J#w and J accepts w */
    return(regular(
          "boolean foo(String x) {"
        + "   if(anbn(x))"
        + "       return(true);"
        + "   else"
        + "       return(simulate(" + s + "));"
        + "}"
    )); }
```

- If s is a string of the form $J\#w$ and Java program J accepts input w, then `foo` accepts all strings. Otherwise, `foo` only accepts strings of the form $\mathtt{a}^n\mathtt{b}^n$.
- In other words, the language of `foo` is regular iff J accepts w.
- If we could decide $REGULAR_J$, we could also decide A_J.
- A_J is not decidable (just like A_{TM}). Therefore $REGULAR$ is not decidable either.

$REGULAR$ is not decidable (TM-1)

- If $REGULAR$ were decidable, then there would be a TM, M_{REG} that decides it.
- We'll show that if we had M_{REG}, we could build another TM, $M_{A_{TM}}$ that would decide A_{TM}.
- When run with input string s, here's what $M_{A_{TM}}$ will do:
  - Compute the description of a TM, M_{foo}:
    If run with input x, M_{foo} will
    - Check to see if x has the form $a^n b^n$ and if so accept.
    - Otherwise, M_{foo} simulates M running with input w.
      - If M accepts w, then M_{foo} accepts x.
      - If M rejects w, then M_{foo} rejects x.
      - If M loops on w, then M_{foo} loops on x.

    Note that $L(M_{foo})$ is regular iff M accepts w.
  - $M_{A_{TM}}$ now runs M_{REG} on the description of M_{foo}.
    - If M_{REG} accepts M_{foo} then $M_{A_{TM}}$ accepts s (i.e. $M\#w$).
    - If M_{REG} rejects M_{foo} then $M_{A_{TM}}$ reject s.
    - If M_{REG} cannot loop on M_{foo} because it was assumed to be a decider.

$REGULAR$ is not decidable (TM-2)

- If we had a TM, M_{REG} that was a decider for the language $REGULAR$,
- Then we could construct a TM, $M_{A_{TM}}$ that would be a decider for A_{TM}.
- We know that A_{TM} is not decidable.
- Thus, we cannot build a decider for $REGULAR$.
- Therefore, $REGULAR$ is not Turing decidable.

Reducing $\overline{A_{TM}}$ to REGULAR

- This time, our $M_{\overline{A_{TM}}}$ will compute th description of M_{bar}.
  - M_{bar} simulates M running with input w.
    - If M accepts w, then M_{bar} checks to see if x has the form $a^n b^n$.
      - If x has the form $a^n b^n$, M_{bar} accepts x.
      - Otherwise, M_{bar} rejects x.
    - If M rejects w, then M_{bar} rejects x.
    - If M loops on w, then M_{bar} loops on x.

  $L(M_{bar})$ is regular iff M rejects w.
- If we had a TM, M_{REG} that was a decider for the language $REGULAR$,
  - Then we could construct a TM, $M_{\overline{A_{TM}}}$ that would be a decider for $\overline{A_{TM}}$.
  - We know that $\overline{A_{TM}}$ is not decidable.
  - Thus, we cannot build a decider for $REGULAR$.
  - Therefore, $REGULAR$ is not Turing decidable.

$\overline{A_{TM}}$ to REGULAR in Java

- This time, we write

```
boolean aJbar(String s) { /* return true s = J#w and J rejects w */
    return(regular(
            "boolean bar(String x) {"
        + "   if(simulate(" + s + "))"
        + "       if(anbn(x)) return(true);"
        + "       else return(false);"
        + "   else return(false);"
        + "}"
    )); }
```

- If s is a string of the form $J\#w$ and Java program J accepts input w, then bar accepts strings of the form $\mathtt{a}^n\mathtt{b}^n$. Otherwise, bar rejects or loops on all strings.
- In other words, the language of bar is regular iff J does not accept w.
- If we could decide $REGULAR_J$, we could also decide $\overline{A_J}$ which is equivalent to $\overline{A_{TM}}$.
- $\overline{A_{TM}}$ is not decidable. Therefore $REGULAR$ is not decidable either.

How hard is $REGULAR$?

- We cannot reduce $REGULAR$ to A_{TM}. Why not?
  - If we could, then we could reduce $\overline{A_{TM}}$ to A_{TM} – we've shown that we can reduce $\overline{A_{TM}}$ to $REGULAR$.
  - Then, we could build a decider for A_{TM}:
    Given an input $M\#w$, we could run a recognizer for A_{TM} and a recognizer for $\overline{A_{TM}}$ until one accepts. If the recognizer for A_{TM} accepts, we accept, and if the recognizer for $\overline{A_{TM}}$ accepts, then we reject.
  - But, A_{TM} is not decidable.
  - Therefore, we can't reduce $REGULAR$ to A_{TM}.
- We cannot reduce $REGULAR$ to $\overline{A_{TM}}$ either.
  The proof has the same form as the proof above.

Quantifying Decidability

- <ExtraCredit>
- A language, A, is Turing decidable iff there is a TM that decides it.
  - Examples: any regular or context free language, testing for primality, any NP-complete problem, anything for which you have an algorithm.
- A language, B, is Turing recognizable iff there is a Turing decidable langauge A such that:

$$B = \{s \mid \exists x.\ s \dagger x \in A\}$$

  - Example, $HALT$. Let

$$A = \{M\#w \dagger n \mid M \text{ describes a Turing machine, } w \text{ describes a string, and } n \text{ is the binary representation of an integer, such that TM } M \text{ halts within } n \text{ steps when run with input } w.\}$$

  A is decidable (see midterm 2). Thus, $HALT$ is Turing recognizable.
  - In this case, we used the existential quantifier to say that if M accepts w, then there must be some integer n such that M accepts w after at most n steps. This can be verified by simulating M for at most n steps.

Quantifying Decidability

- Let $accept(M, w, n)$ denote that TM M accepts w after at most n steps.
- A language, E, is Turing co-recognizable iff there is a Turing decidable langauge A such that:

$$B = \{s \mid \forall x.\ s \dagger x \in A\}$$

  - Examples

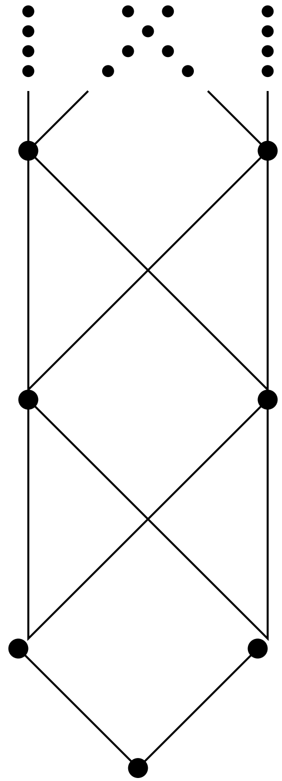
$$\begin{aligned}
\overline{A_{TM}} &= \{M\#w \mid \neg\exists n.\ accept(M, w, n)\} \\
&= \{M\#w \mid \forall n.\ \neg accept(M, w, n)\}
\end{aligned}$$

$$\overline{E_{TM}} = \{M \mid \forall w, n.\ \neg accept(M, w, n)\}$$

- What about $REGULAR$?

$$\begin{aligned}
REGULAR = \{M\#w \mid \exists D.\ \forall w.\ & \\
& DFArecognize(D, w) \Rightarrow \exists n.\ accept(M, w, n) \\
\wedge\ & \neg DFArecognize(D, w) \Rightarrow \forall n.\ \neg accept(M, w, n)
\end{aligned}$$

The Arithmetic Hierarchy



$\Sigma_3 = \exists x_3.\ \forall x_2.\ \exists x_1.\ p(s, x_1, x_2, x_3)$
E.g. $REGULAR$

$\Pi_3 = \forall x_3.\ \exists x_2.\ \forall x_1.\ p(s, x_1, x_2, x_3)$

$\Sigma_2 = \exists x_2.\ \forall x_1.\ p(s, x_1, x_2)$
E.g. $FINITE$

$\Pi_2 = \forall x_2.\ \exists x_1.\ p(s, x_1, x_2)$
E.g. $TOTAL$

$\Sigma_1 = \exists x_1.\ p(s, x_1)$
E.g. $HALT$, A_{TM}

$\Pi_1 = \forall x_1.\ p(s, x_1)$
E.g. E_{TM}

Decidable: $p(s)$

- </ExtraCredit>

This coming week (and beyond)

- Reading
  - Today: Sipser 5.1
  - Nov. 10 (Monday): Sipser 5.2
  - Nov. 12 (Wednesday): Sipser 5.3
  - Nov. 14 (A week from today): Sipser 7.1
- Homework
  - Today: HW 9 goes out.
  - Nov. 10 (Monday): HW 8 due.
  - Nov. 14 (a week from today): HW 10 goes out.
  - Nov. 17 (a week from Monday): HW 9 due.